Separable Ordinary Differential Equations – (Brief) Review

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1. Definition.

   A function $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called *separable* if there exist functions f, g such that

   $$F(x,y) = f(x)g(y)$$

   for all $(x,y) \in D$.

2. Example.

   Let

   $$F(x,y) = xy - xy^3 + x^2y - x^2y^3$$

   Then F is separable since:

   $$F(x,y) = (x + x^2)(y - y^3)$$

3. Claim.

   Let $\Omega \subseteq \mathbb{R}^2$ be open and convex. If F, D_1F, D_2F D_1D_2F are continuous on Ω and $F \neq 0$ on Ω, then following are equivalent:

   (a) F is separable.

   (b) $FD_1D_2F = D_1FD_2F$.

   Proof.

   Suppose (b) holds. We can assume that $F > 0$. Then

   $$D_2\frac{D_1F}{F} = \frac{FD_2D_1F - D_1FD_2F}{F^2}.$$

   The conditions of the theorem guarantee that

   $$D_1D_2F = D_2D_1F.$$

   Hence on Ω,

   $$\begin{aligned} D_2D_1 \log F &= D_2\frac{D_1F}{F} \\ &= 0. \end{aligned}$$

   It follows that $D_1 \log F$ is independent of y. Therefore there exists a function g such that:

   $$D_1 \log F(x,y) = g(x).$$

   Therefore,

   $$\log F(x,y) = \int g(x)dx + h(y).$$

   for some function h and so

   $$F(x,y) = e^{\int g(x)dx}e^{h(y)},$$

   which is separable. The converse is trivial.

4. Definition.

   A differential equation $y' = F(x,y)$ is called *separable* if F is a separable function.

5. Trivial claim.

   If $g(y_0) = 0$ then the separable IVP:

   $$y' = \frac{g(y)}{f(x)}; \quad y(x_0) = y_0 \qquad (1)$$

   has a (not necessarily unique) solution.

6. Claim.

   If f and g are continuous and $g(y_0) \neq 0$ and $f(x_0) \neq 0$ then in a neighborhood of (x_0, y_0) there exists a unique solution ϕ of the IVP (1) given implicitly by

   $$\int_{y_0}^{\phi(x)} \frac{1}{g(\eta)}d\eta = \int_{x_0}^{x} \frac{1}{f(\xi)}d\xi \qquad (2)$$

   Proof.

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Let
$$G(y) = \int_{y_0}^{y} \frac{1}{g(\eta)} d\eta$$
and
$$F(y) = \int_{x_0}^{x} \frac{1}{f(\xi)} d\xi$$
Since $g(y_0) \neq 0$, Either $G' > 0$ of $G' < 0$ in some in a neighborhood of y_0. It follows that in this neighborhood G has an inverse G^{-1}. Let
$$\phi(x) = G^{-1} \circ F(x)$$
Notice
$$\begin{aligned} \phi'(x) &= G^{-1\prime}(F(x))F'(x) \\ &= \frac{F'(x)}{G'(G^{-1} \circ F(x)})} \\ &= \frac{g(\phi(x)}{f(x)} \end{aligned}$$
Moreover,
$$\begin{aligned} \phi(x_0) &= G^{-1} \circ F(x_0) \\ &= G^{-1}(0) \\ &= y_0 \end{aligned}$$
Thus ϕ is indeed a (local) solution of the IVP (1) If α is any solution, since $g(y_0) \neq 0$ and $f(x_0) \neq 0$, we have in some neighborhood of x_0 that
$$\int_{x_0}^{x} \frac{\alpha'(\xi)}{g(\alpha(\xi)} d\xi = \int_{x_0}^{x} \frac{1}{f(\xi)} d\xi$$
By the change of variables formula we see that α satisfies
$$\int_{y_0}^{\alpha(x)} \frac{1}{g(\eta)} d\eta = \int_{x_0}^{x} \frac{1}{f(\xi)} d\xi$$
That is,
$$G \circ \alpha = F.$$
Since G is locally invertible, it follows that $\alpha = G^{-1} \circ F = \phi$ in some neighborhood of x_0. Hence ϕ is locally unique in the case $g(y_0) \neq 0$.

7. Remark.

   Let $H(x, y) = G(y) - F(x)$. The solution ϕ given by (2) is a solution of the implicit equation $H(x, y) = 0$. Since $H \in \mathcal{C}^1$ and $\frac{\partial H}{\partial y} \neq 0$ in a neighborhood of (x_0, y_0), the implicit function theorem also guarantees local uniqueness of the solution.

8. Special cases of (2).

   (a) Autonomous ODE's.

   If $x' = v(x); \quad x(t_0) = x_0$, then (2) reduces to
$$\int_{x_0}^{\phi(t)} \frac{1}{v(\xi)} d\xi = t - t_0 \tag{3}$$

   (b) Trivial linear ODE's.

   If $x' = q(t); \quad x(t_0) = x_0$, then (2) can be solved for ϕ and
$$\phi(t) = x_0 + \int_{t_0}^{t} q(\tau) d\tau.$$

Math 4009 - Homework: Separable Equations

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1. A function $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called *separable on* D if there exist functions f, g such that $F(x, y) = f(x)g(y)$ for all $(x, y) \in D$.

   (a) Let $h(x, y) = xy - xy^3 + x^2y - x^2y^3$. Show that h separable on $\mathbb{R}^2$.

   (b) Let $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be separable. Suppose that the partial derivatives D_1F, D_2F D_1D_2F exist on D. Show that $FD_1D_2F = D_1FD_2F$.

2. (a) Define *separable* differential equation.

   (b) Solve the initial value problem $y' = -x/y, \quad y(1) = -\sqrt{3}$.

   (c) Sketch all solutions of the differential equation $y' = -x/y$.

3. Find solutions in implicit form of the following differential equations:

   (a) $y' = \dfrac{(2x + 1)y}{\cos \ln y}$.

   (b) $xy' = (1 - y^2)^{1/2}$.

   (c) $y' = \dfrac{2 - e^x}{1 + y}$

4. Solve the initial value problem $y' = \dfrac{3x^2}{3y^2 - 4}, \quad y(1) = 0$ and determine the maximal interval in which the solution is valid.

5. A stone of mass m is dropped from a bridge and falls for a time T. If the motion obeys the equation $m\ddot{x} = gm - c\dot{x}^2$, determine the height of the bridge in terms of the c, m, g and T

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