Forced Oscillator - Steady State Solution by Fourier Transform

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1. Forced damped oscillator equation:

$$x'' + 2\gamma x' + \omega_0^2 x = F(t) \tag{1}$$

2. Let $x(t)$ denote a solution of (1). Taking the Fourier transform (angular frequency version) produces:

$$\hat{x}(\omega) = \frac{\hat{F}(\omega)}{p(\omega)} \tag{2}$$

where

$$p(\omega) = -\omega^2 + 2i\omega\gamma + \omega_0^2$$

3. Inverting (2):

$$x(t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega t}\hat{F}(\omega)}{p(\omega)}\,d\omega \tag{3}$$

4. Since this is a time invariant linear system the solution can be written as the convolution of the impulse response h with F.

$$\begin{aligned} x(t) &= h * F(t) \\ &= \int\limits_{\mathbb{R}} h(t-\tau)F(\tau)d\tau \end{aligned}$$

5. Claim. The impulse response function h is given by

$$h(t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega t}}{p(\omega)}d\omega$$

Proof 1.

Replace F by a δ function in (3) and extract h from the resulting convolution.

Proof 2. [Using Fubini]

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega}\hat{F}(\omega)}{p(\omega)}\,d\omega \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega t}}{p(\omega)}\left(\int\limits_{\mathbb{R}} e^{-i\omega\tau}F(\tau)d\tau\right)d\omega \\ &= \frac{1}{2\pi}\iint\limits_{\mathbb{R}\times\mathbb{R}} \frac{e^{i\omega(t-\tau)}}{p(\omega)}F(\tau)d(\tau,\omega) \\ &= \int\limits_{\mathbb{R}} h(t-\tau)F(\tau)d\tau \end{aligned}$$

where h is given by

$$h(t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega}}{p(\omega)}d\omega$$

Explicit computation of h in particular cases is aided by:

6. Jordan's Lemma

Let C_R be the semicircle in the upper half-plane of radius R and center the origin and f has, at worst, a finite number of poles. If $f(z) \to 0$ as $|z| \to \infty$ and $\lambda > 0$ then $\int_{C_R} f(z)e^{i\lambda z}dz \to 0$ as $R \to \infty$. If $\lambda < 0$ the result holds for the corresponding semicircle C_R' in the lower half plane.

7. Claim. If $0 < \gamma < \omega_0$ (small damping) then the oscillator (1) has impulse response:

$$h(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{\beta}e^{-\gamma t}\sin\beta t & \text{if } t \geq 0. \end{cases}$$

where $\beta = \sqrt{\omega_0^2 - \gamma^2}$.

Proof. Notice that p has zeros in the upper half plane:

$$\omega_\pm = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

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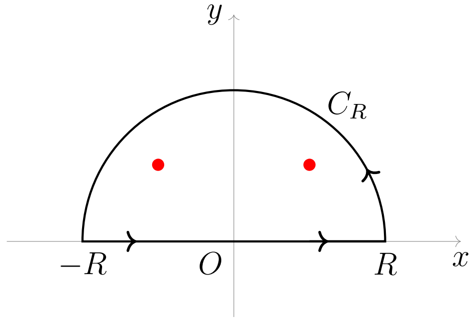
and so

$$h(t) = -\frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{i\omega t}}{(\omega-\omega_+)(\omega-\omega_-)}d\omega$$

Let C (shown below) be the positively oriented closed path comprising the segment $[-R,R]$ and followed by C_R.



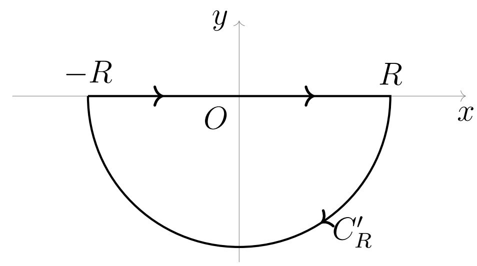
Notice

$$\int_C = \int_{-R}^{R} + \int_{C_R}$$

for all R. If $t \geq 0$, letting $R \to \infty$ and invoking Jordans lemma and the residue theorem yields:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int\limits_{C} \frac{e^{i\omega t}}{(\omega_+-\omega)(\omega-\omega_-)}d\omega \\ &= -i\left[\frac{e^{i\omega_+ t}}{\omega_+-\omega_-} - \frac{e^{i\omega_- t}}{\omega_+-\omega_-}\right] \\ &= -\frac{ie^{-\gamma t}}{2\beta}\left[e^{i\beta t} - e^{-i\beta t}\right] \\ &= -\frac{ie^{-\gamma t}}{2\beta}\left[2i\sin(\beta t)\right] \\ &= \frac{1}{\beta}e^{-\gamma t}\sin\beta t \end{aligned}$$

Now, let C' be the oriented path in the lower half plane comprising the segment $[-R,R]$ and followed by the semicircle C'_R.



By analicity,

$$\int\limits_{\mathbf{C'}} \frac{e^{i\omega t}}{p(\omega)}d\omega = 0.$$

Hence, using Jordan's lemma

$$\begin{aligned} \int\limits_{\mathbb{R}} \frac{e^{i\omega}}{p(\omega)}d\omega &= -\lim_{R\to\infty}\int\limits_{\mathbf{C'_R}} \frac{e^{i\omega t}}{p(\omega)}d\omega \\ &= 0 \end{aligned}$$

It follows that $h(t) = 0$ for $t < 0$.