Existence and Uniquness – the Bielecki Renorming Technique

Philip Pennance[1] – 1974.

1. Claim

   Let $\Omega \subseteq \mathbb{R}^n$ a region and

   $$D = [t_0, t_0 + h] \times \Omega.$$

   Let $f : D \to \mathbb{R}^n$.

   If f is continuous on D and $\exists L > 0$ such that for all $(t, x), (t, \hat{x}) \in D$

   $$|f(t,x) - f(t,\hat{x})| < L|x - \hat{x}|$$

   (where $| \ |$ is the $\mathbb{R}^n$ norm), then

   $$x' = f(t,x)$$

   has a unique solution for $t_0 \leq t \leq t + h$.

   Proof. (Bielecki, 1965.)

   Let

   $$X = \mathcal{C}[t_0, t_0 + h]$$

   be the vector space of continuous functions from $[t_0, t_0 + h]$ to $\mathbb{R}^n$.

   For each $k > L$, define (a weighted norm) $\| \ \| : X \to \mathbb{R}$ by

   $$\|x\| = \sup_{[t_0, t_{0+h}]} \left(|x(t)| e^{-k(t-t_0)} \right).$$

   Then $\langle X, \| \ \| \rangle$ is a Banach space. Define the *Picard mapping* $T : X \to X$ by

   $$T(x) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

   Then,

   $$\begin{aligned} & |T(x_1) - T(x_2)| \\ & \leq \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_2(s))|\, ds \\ & \leq L \int_{t_0}^{t} |x_1(s) - x_2(s)|\, ds \\ & = L \int_{t_0}^{t} e^{-k(s-t_0)} |x_1(s) - x_2(s)| e^{k(s-t_0)}\, ds \\ & = L \|x_1 - x_2\| \int_{t_0}^{t} e^{k(s-t_0)}\, ds \\ & = \frac{L}{K} \|x_1 - x_2\| e^{k(t-t_0)} \end{aligned}$$

   It follows that

   $$\|T(x_1) - T(x_2\| \leq \frac{L}{K} \|x_1 - x_2\|.$$

   Since $0 < L < K$, T is a contraction mapping. By the Banach fixed point theorem, $\exists$ a unique $x \in X$ such that $Tx = x$. I.e.,

   $$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)\, ds.$$

   This is the required unique solution.

[1] https://pennance.us