Linear Systems and Fundamental Matrices

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1. Claim.

   Let
   $$A : [a,b] \to \mathbb{R}^{n\times n}$$
   be continuous. Let Φ be a solution matrix ($\Phi \in \mathrm{SM}$) of the ODE
   $$X' = AX. \tag{1}$$
   The following are equivalent.

   (a) Φ is a fundamental matrix.

   (b) $\det \Phi(t) \neq 0$ for all $t \in [a,b]$.

   Proof.

   Suppose that Φ is a fundamental matrix. If ψ is a solution of (1), there exists a unique $c \in \mathbb{R}^n$ such that
   $$\psi(t) = \Phi(t)\mathbf{c}.$$
   Hence,
   $$\psi(t_0) = \Phi(t_0)\mathbf{c}.$$
   is a linear system in $\mathbb{R}^n$ with a unique solution. This means that
   $$\det \Phi(t_0) \neq 0.$$
   By the Abel-Jacobi-Liouville identity $\det \Phi(t)$ is non vanishing.

   Conversely, suppose that $\det \Phi$ is non vanishing in $[a,b]$. Since $\Phi \in \mathrm{SM}$, to show that $\Phi \in \mathrm{FM}$, it suffices to show its the columns are linearly independent.
   $$\begin{aligned} &\sum_{i=1}^{n} c_i \Phi^i(t) = 0_{\mathbb{R}^n}; \quad t \in [a,b] \\ \implies &\sum_{i=1}^{n} c_i \Phi^i(t_0) = 0_{\mathbb{R}^n} \\ \implies &\Phi(t_0)\mathbf{c} = 0_{\mathbb{R}^n} \end{aligned}$$
   Since $\det \Phi(t_0) \neq 0$ it must be that $\mathbf{c}$ is the zero vector.

2. Corollary.

   For any invertible matrix $B \in \mathbb{R}^{n\times n}$
   $$\Phi \in \mathrm{FM} \implies \Phi B \in \mathrm{FM}\,.$$
   Proof.
   $$\begin{aligned} (\Phi B)' &= \Phi' B \\ &= (A\Phi)B \\ &= A(\Phi B). \end{aligned}$$
   Hence $\Phi B \in \mathrm{SM}$. Moreover,
   $$\det(\Phi B) = \det \Phi \det B$$
   is non vanishing. By the preceeding result, it follows that
   $$\Phi B \in \mathrm{FM}\,.$$

3. Corollary.

   Let $\Psi, \Phi \in \mathrm{FM}$. Then there exists an invertible matrix C such that
   $$\Phi = \Psi C.$$
   Proof.

   Each Φ^i is a solution of the homogeneous equation $X' = AX.$. Since the columns of Ψ form a basis for the solution space, there exist a vector C^i such that
   $$\Phi^i = \Psi C^i.$$
   If C is the matrix with columns $C^1, C^2, \ldots, C^n$ then
   $$\Phi = \Psi C.$$
   Since $\Phi, \Psi \in \mathrm{FM}$ they have non vanishing determinants and so $\det C \neq 0$.

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4. Let $A \in \mathbb{R}^{n\times n}$. The matrix e^{tA} is defined by
$$e^{tA} = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!}.$$

5. Claim.

   $\Phi(t) = e^{tA}$ is the unique fundamental matrix such that
$$\Phi(0) = I.$$

   Proof.
$$\begin{aligned}\frac{d}{dt}e^{tA} &= A\sum_{i=1}^{\infty}\frac{t^{i-1}A^{i-1}}{(i-1)!}\\ &= A\sum_{i=0}^{\infty}\frac{t^i A^i}{i!}\\ &= Ae^{tA}.\end{aligned}$$

   and so $e^{tA} \in \mathrm{SM}$. Moreover, $\Phi(0) = I$ It follows using the Abel-Jacobi-Liouville identity that $\det\Phi$ is never zero. Hence $\Phi \in \mathrm{FM}$.

6. Claim .

   Let Ψ be any fundamental matrix, then
$$e^{tA} = \Psi(t)\Psi^{-1}(0).$$

   Proof.

   Since $e^{tA} \in \mathrm{FM}$, by corollary (3), there exists $C \in \mathbb{R}^{n\times n}$ invertible such that
$$e^{tA} = \Psi C.$$

   Putting $t = 0$ yields
$$C = \Psi^{-1}(0)$$

   and so
$$e^{tA} = \Psi(t)\Psi^{-1}(0).$$

7. Claim.

   Suppose that $A \in \mathbb{R}^{n\times n}$ has n independent eigenvectors $V^1, V^2, \ldots, V^n$ with eigenvalues $\lambda_1.\lambda_2, \ldots, \lambda_n$ respectively. Let Ψ be the matrix with columns given by
$$\Psi^i = e^{\lambda_i t}V^i.$$

   Then $\Psi \in \mathrm{FM}$.

   Proof.

   Since
$$AV^i = \lambda_i V^i.$$
$$\begin{aligned}\frac{d}{dt}\Psi^{i\prime}(t) &= \lambda_i e^{\lambda_i t}V^i\\ &= e^{\lambda_i t}AV^i\\ &= A\Psi^i(t).\end{aligned}$$

   and so $\Psi \in \mathrm{SM}$. Since the V^i are independent, $\Psi \in \mathrm{FM}$.

8. Example.

   Let
$$A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

   A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors
$$V^1 = \begin{bmatrix}1\\ 1\end{bmatrix}, \quad V^2 = \begin{bmatrix}1\\ -1\end{bmatrix},$$

   By claim (7),
$$\begin{aligned}\Psi(t) &= \begin{bmatrix}e^{\lambda_1 t}V^1 & | & e^{\lambda_2 t}V^2\end{bmatrix}\\ &= \begin{bmatrix}e^t & e^{-t}\\ e^t & -e^{-t}\end{bmatrix}\end{aligned}$$

   is a fundamental matrix and
$$\Psi^{-1} = -\frac{1}{2}\begin{bmatrix}-e^{-t} & -e^{-t}\\ -e^t & e^t\end{bmatrix}.$$

   Hence,
$$\begin{aligned}e^{tA} &= \Psi(t)\Psi^{-1}(0)\\ &= \frac{1}{2}\begin{bmatrix}e^t & e^{-t}\\ e^t & -e^{-t}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}e^t + e^{-t} & e^t - e^{-t}\\ e^t - e^{-t} & e^t + e^{-t}\end{bmatrix}.\end{aligned}$$

9. Claim.

Suppose that $A \in \mathbb{R}^{n\times n}$ has n independent eigenvectors $V^1, V^2, \dots, V^n$ with eigenvalues $\lambda_1.\lambda_2, \dots, \lambda_n$ respectively. Then

$$V^{-1}AV = \Lambda$$

where V is the matriz with columns $V^1, V^2, \dots, V^n$ and Λ is the diagonal matrix

$$\begin{aligned}\Lambda(i,j) &= \lambda_i \delta(i,j)\\ &= \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}.\end{aligned}$$

Proof.

$$\begin{aligned}(V^{-1}AV)(i,j) &= V_i^{-1}(AV)^j\\ &= V_i^{-1}(AV^j)\\ &= \lambda_j V_i^{-1}V^j\\ &= \lambda_j (V^{-1}V)(i,j)\\ &= \lambda_j \delta(i,j)\end{aligned}$$

10. Corollary

Suppose that $A \in \mathbb{R}^{n\times n}$ has n independent eigenvectors $V^1, V^2, \dots, V^n$ with eigenvalues $\lambda_1.\lambda_2, \dots, \lambda_n$ respectively. Then

$$\begin{aligned}e^{tA} &= V e^{t\Lambda} V^{-1}\\ &= V \begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix} V^{-1}.\end{aligned}$$

Proof.

$$\begin{aligned}e^{tA} &= \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i\\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!} (V\Lambda V^{-1})^i\\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!} (V\Lambda^i V^{-1}\\ &= V\left(\sum_{i=0}^{\infty} \frac{t^i}{i!}\Lambda^i\right) V^{-1}\\ &= V e^{t\Lambda} V^{-1}\\ &= V \begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix} V^{-1}.\end{aligned}$$

11. Example.

$$A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors

$$V^1 = \begin{bmatrix}1\\ 1\end{bmatrix}, \quad V^2 = \begin{bmatrix}1\\ -1\end{bmatrix},$$

Let

$$V = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$V^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

Then

$$\begin{aligned}e^{tA} &= V \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix} V^{-1}\\ &= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}e^t + e^{-t} & e^t - e^{-t}\\ e^t - e^{-t} & e^t + e^{-t}\end{bmatrix}\end{aligned}$$

Remark.

Let $\eta \in \mathbb{R}^n$. Since e^{tA} is a fundamental matrix, the initial value problem

$$\begin{cases} X' = AX, \\ X(0) = \eta. \end{cases}$$

has unique solution

$$X(t) = e^{tA}\eta.$$

12. Claim.

Let

$$A : [a, b] \to \mathbb{R}^{n\times n}$$

and

$$\mathbf{g} : [a, b] \to \mathbb{R}^n$$

be continuous. If Φ is a fundamental matrix of the homogeneous equation, then the IVP

$$\begin{cases} X' = A(t)X + \mathbf{g}(t) \\ Xt_0) = 0_{\mathbb{R}^n}. \end{cases} \tag{2}$$

has particular solution

$$\varphi_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{g}(s)\, ds.$$

Proof.

As in the method of variation of parameters we seek a function $v(t)$ such that

$$\varphi_p = \Phi(t)v(t)$$

is a particular solution.

Substitution this into the differential equation and noting that

$$\Phi' = A\Phi,$$

yields

$$\Phi v' = \mathbf{g}.$$

Hence,

$$v(t) = \int \Phi^{-1}(t)\mathbf{g}(t)\, dt + C.$$

Noting that $\varphi_{\mathbf{p}}(t_0) = 0$, it follows, that

$$\begin{aligned} \varphi_p(t) &= \Phi(t)v(t) \\ &= \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{g}(s)\, ds. \end{aligned}$$

13. Corollary.

Let $\eta \in \mathbb{R}^n$. To find a particular solution φ_p such that $\varphi_p(t_0) = \eta$, let $\mathbf{c}$ be the unique vector such that

$$\Phi(t_0)\mathbf{c} = \eta.$$

Since

$$\Phi(t)\mathbf{c} = \Phi(t)\Phi^{-1}(t_0)\eta$$

is a solution of the homogeneous equation, it follows that

$$\begin{aligned} \varphi_p(t) &= \Phi(t)\Phi^{-1}(t_0)\eta \\ &\quad + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)g(s)\, ds \end{aligned}$$

is the unique solution, satisfying the initial condition $X(t_0) \ = \eta$.

$$\Phi(t)\Phi^{-1}(0) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \Phi(t) \int_0^t \Phi^{-1}(s) \begin{bmatrix} \sin s \\ \cos s \end{bmatrix} ds'$$

14. Example.

Solve the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

$$x_1(0) = 1, \quad x_2(0) = -1,$$

given that

$$\Phi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

is a fundamental matrix.

Solution.

$$\Phi^{-1}(t) = e^{-4t} \begin{bmatrix} e^{2t} & -te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

$$\Phi^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The unique solution of the IVP can now be easily computed from

$$\varphi(t) = \Phi(t)\Phi^{-1}(0)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{bmatrix} \sin s \\ \cos s \end{bmatrix} ds.$$