Exact Differential Equations, Integrating Facors and Vector Fields

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1. Introduction.

   We examine the geometric significance of integrating factors, when used in the solution of the non-autonomous differential equation determined by a vector field in a region of $\mathbb{R}^2$.

2. Definition.

   Let $U \subseteq \mathbb{R}^2$ be open. By a *vector field on U* we mean a function $V : U \to \mathbb{R}^2$.

3. Definition.

   Let $V((x, y) = (v_1(x, y), v_2(x, y))$ be a vector field. The expression:

   $$\begin{cases} \dfrac{dx}{dt} = v_1(x, y) \\ \dfrac{dy}{dt} = v_2(x, y) \end{cases} \tag{1}$$

   is called the *autonomous system* of differential equations determined by V.

   If v_1 is non vanishing,

   $$\frac{dy}{dx} = \frac{v_2(x, y)}{v_1(x, y} \tag{2}$$

   is called the *non-autonomous differential equation determined by V*.

4. Definition.

   A solution of (1) is a curve $\alpha : I \to \mathbb{R}^2$ defined on an interval I such that

   $$\begin{cases} \alpha_1'(t) = v_1(\alpha_1(t), \alpha_2(t)) \\ \alpha_2'(t) = v_2(\alpha_1(t), \alpha_2(t)) \end{cases} \tag{3}$$

5. Example:

   Let $V(x, y) = (-y, x)$. The autonomous system determined by V is:

   $$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = \quad x \end{cases} \tag{4}$$

   or in matrix form

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   By inspection $\alpha(t) = (\cos t, \sin t)$ is one solution[2] of (4).

6. Application.

   Along a regular adiabatic process, the temperature θ volume V of a homogeneous thermodynamic system is known to be a function of the volume V. satisfying

   $$\frac{d\theta}{dV} = -\frac{\Lambda(\theta, V)}{K(\theta, V)}.$$

   The functions K and Λ are called the *specific heat* and *latent heat* respectively. Thus, the non-autonomous differential equation of an adiabat is determined by the vector field $V = (-\Lambda, K)$. (See exercise 7.)

7. Claim.

   Let $V = (v_1, v_2)$ be a vector field with v_1 and v_2 continuous, and v_1 non vanishing. If

   $$\alpha(t) = (\alpha_1(t), \alpha_2(t)), \quad t \in I$$

   is a solution of the autonomous system (1), then

   (a) $\alpha_1(t)$ is invertible.

   (b) The trajectory (image) of α is the graph of the function ϕ given by

   $$\phi(x) = \alpha_2 \circ \alpha_1^{-1}(x) \tag{5}$$

   (c) ϕ is a solution of the *non-autonomous differential equation determined by V* viz:

   $$\frac{dy}{dx} = \frac{v_2(x, y)}{v_1(x, y} \tag{6}$$

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[2] Remark: The solution may be determined using the exponential function of the matrix.

$$\begin{tikzcd} G \arrow[r, two heads, "\alpha_1"] \arrow[d, "\alpha_2"'] & \operatorname{Im}\alpha_1 \arrow[dl, dashed, "\phi"] \\ \mathbb{R} & \end{tikzcd}$$

Proof:

Since α is a solution of (1),

$$\alpha_1'(t) = v_1(\alpha_1(t), \alpha_2(t)).$$

Since v_1 is continuous and non-vanishing $\alpha_1'(t)$ must be everywhere positive or everywhere negative on I. In either case, we deduce that α_1 is a bijection from I onto its image. Let $\phi(x) = \alpha_2 \circ \alpha_1^{-1}(x)$. Then using (3) and (5)

$$\begin{aligned} \phi'(x) &= \alpha_2'(\alpha_1^{-1}(x))({\alpha_1}^{-1})'(x) \\ &= \frac{v_2(x,\phi(x))}{v_1(x,\phi(x))} \end{aligned} \tag{7}$$

Thus ϕ is a solution of (6).

Let G_ϕ be the graph of ϕ. We claim $\operatorname{Im}\alpha = G_\phi$. Let $(x,y) \in \operatorname{Im}\alpha$. Then there exists $\tau \in I$ such that $(x,y) = (\alpha_1(\tau), \alpha_2(\tau))$. Therefore $\tau = \alpha_1^{-1}(x)$ and so $y = \alpha_2 \circ {\alpha_1}^{-1}(x) = \phi(x)$. Hence $(x,y) \subseteq G_\phi$. The proof of the converse inclusion is similar.

8. Corollary

   Let U be an open subset of $\mathbb{R}^2$. Let $q : U \to \mathbb{R}$ be continuous and

   $$V(x,y) = (v_1(x,y), v_2(x,y))$$

   a vector field on U. If qv_1 is non vanishing on U and

   $$\alpha(t) = (\alpha_1(t), \alpha_2(t))$$

   is a solution of

   $$\begin{aligned} \frac{dx}{dt} &= q(x,y)v_1(x,y) \\ \frac{dy}{dt} &= q(x,y)v_2(x,y) \end{aligned} \tag{8}$$

   then $\phi = \alpha_2 \circ \alpha_1^{-1}$ is a solution of the non autonomous differential equation determined by V.

   Proof.

   By claim (6b), the solution ϕ is a solution of the non autonomous equation associated with vector field $qV = (qv_1, qv_2)$, which (since q cancels) coincides with the non autonomous differential equation determined by V.

9. Remark.

   Notice that the effect of the function q is to change the speed of the trajectories of the autonomous system (1) but not their images. Thus, solutions of the associated non autonomous equation (6) are uneffected by q.

10. Definition.

    A choice of q that renders (6) immediately integrable is called an *integrating factor*.

11. Example:

    Consider again the vector field $V(x,y) = (-y, x)$. The non-autonomous differential equation determined by V takes the separable form:

    $$\frac{dy}{dx} = -\frac{x}{y} \tag{9}$$

    Suppose we have an the initial condition, $y(x_0) = y_0$ with $y_0 < 0$. Take as integrating factor

    $$q(x,y) = \frac{-1}{xy}.$$

    The autonomous system determined by the vector field qV is:

    $$\begin{aligned} \frac{dx}{dt} &= \frac{1}{x} \\ \frac{dy}{dt} &= -\frac{1}{y} \end{aligned} \tag{10}$$

    Near (x_0, y_0), this system of two autonomous equations in one dimension has solution $\alpha(t) = (\alpha_1(t), \alpha_2(t)$ given by:

    $$\begin{aligned} \int_{x_0}^{\alpha_1(t)} \xi d\xi &= \int_{y_0}^{\alpha_2(t)} -\eta d\eta \\ &= t - t_0. \end{aligned} \tag{11}$$

    which yields:

    $$\alpha_1^{-1}(x) = t_0 + \frac{x^2 - {x_0}^2}{2}.$$

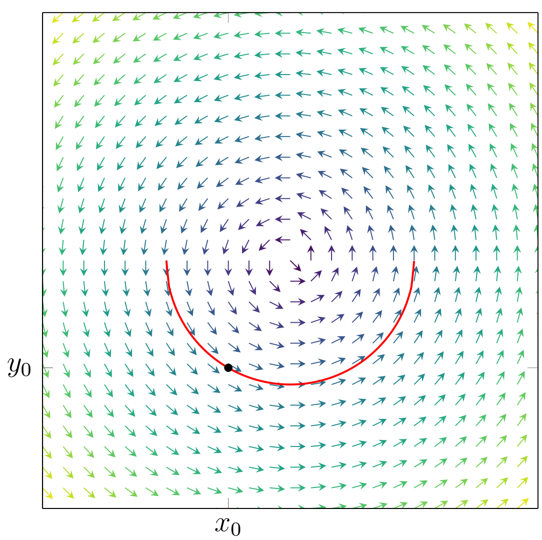
    $$\alpha_2(t) = \pm\sqrt{{y_0}^2 - 2(t - t_0)}.$$

Since $y_0 < 0$ the negative sign applies. By (5) above a solution of the non autonomous differential equation (9) is

$$\begin{aligned} \phi(x) &= \alpha_2 \circ \alpha_1^{-1}(x) \\ &= -\sqrt{{x_0}^2 + {y_0}^2 - x^2}. \end{aligned} \tag{12}$$

The maximal solution of the IVP is a semicircle in the lower half plane, which is in agreement with the fact that trajectories of the autonomous system (4) have the form $\alpha(t) = (r\cos t, r\sin t)$.

Vector Field $V = [-y, x]$



12. Definition.

Let $D \subseteq \mathbb{R}^2$ and

$$V(x,y) = (v_1(x,y), v_2(x,y))$$

be a vector field on D. A function $\Phi : D \to \mathbb{R}$ such that $\nabla\Phi = V$ is called a *potential function* for V. A level curve of V is called an *equipotential*. A vector field is called *exact* if it admits a potential function.

13. Definition.

If $V(x,y) = (v_1(x,y), v_2(x,y))$ is a vector field on an open set U, the vector field determined by:

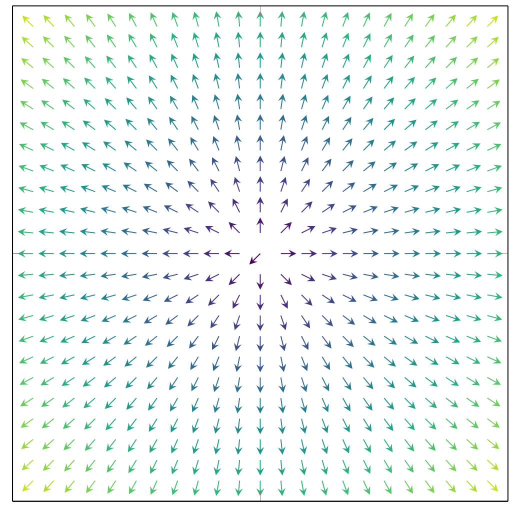
$$V^{\perp}(x,y) = (-v_2(x,y), v_1(x,y))$$

will be called the vector field perpendicular to V.

14. Example.

If $V(x,y) = (-y, x)$, then $V^{\perp}(x,y) = (x,y)$.

Vector Field $V^{\perp} = [x, y]$



15. Claim.

Let V is vector field on $D \subseteq \mathbb{R}^2$ with potential function Φ. Let I be an interval and $\alpha : I \to D$ be a solution of the autonomous system determined by $V^{\perp}$:

$$\begin{aligned} \frac{dx}{dt} &= -v_2(x,y) \\ \frac{dy}{dt} &= v_1(x,y), \end{aligned} \tag{13}$$

then

(a) α' is perpendicular to V.

(b) $\operatorname{Im}\alpha$ is an equipotential curve.

(c) If v_2 is non vanishing, the solutions of the non autonomous equation

$$\frac{dy}{dx} = -\frac{v_1(x,y)}{v_2(x,y} \tag{14}$$

are equipotententials of Φ.

(d) If v_2 is non vanishing, then any equipotential ϕ is a solution of (14)

then $\Phi(\alpha(t))$ is constant on I.

Proof.

$$\begin{aligned} \frac{d\Phi(\alpha(t))}{dt} &= \langle \nabla\Phi(\alpha(t), \alpha'(t)\rangle \\ &= \langle (v_1, v_2), (-v_2, v_1)\rangle \\ &= 0. \end{aligned} \tag{15}$$

from which (a) and (b) follow. The proof of (c) is similar to that of equation (6). Finally, let ϕ is an equipotential, viz,

$$\frac{d}{dx}\Phi(x, \phi(x) = 0.$$

Then, by the chain rule

$$\langle \nabla\Phi(x, \phi(x)), (1, \phi'(x)\rangle = 0.$$

Since $\nabla\Phi = V$, it follows that

$$\frac{d\phi}{dx} = -\frac{v_1(x, \phi)}{v_2(x, \phi(x)}$$

and so ϕ is a solution of (14).

16. Remark.

If V is continuously differentiable and $D_2V \neq 0$, the implicit function theorem guarantees the (local) existence of a function ϕ such that $V(x, \phi(x)) = c$ where c is constant.

17. Example.

Use a potential function to solve

$$\frac{dy}{dx} = -\frac{x}{y} \tag{16}$$

subject to the initial condition, $y(x_0) = y_0$ with $y_0 < 0$.

Notice that the vector field $V = (x, y)$ is exact with potential function

$$\Phi(x, y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

Since (16) is just the non autonomous differential equation determined by $V^\perp = (-y, x)$ the solutions are determined by the equipotentials of Φ. Since the second component of the vector field y must be non vanishing and $y_0 < 0$, maximal solutions are semicircles of the form (12).

18. Remark.

The reason for the term potential derives from physics. If F is a force and $F = -\nabla\Phi$, then the *energy*

$$\frac{1}{2}m|x(t)'|^2 + \Phi(x(t))^2$$

is constant along solutions of the Newton/Euler equation $F = mx''$.

19. Remark.

If a vector field $V = (v_1, v_2)$ is not exact, it may happen that there exists an integrating factor q such that $\hat{V} = (qv_1, qv_2)$ is exact. Since the solutions of the non-autonomous differential equation are unchanged by the introduction of an integrating factor, the equation can also be solved by finding a potential function Φ such that $\nabla\Phi = \hat{V}$ and finding its level curves. This case will be discussed in a subsequent document.

20. Example.

21. Find b such that

$$xy^2 + bx^2y + (x + y)x^2y' = 0 \tag{17}$$

is an exact differential equation. Find the general solution in implicit form.

Solution.

Let

$$\begin{aligned} M &= xy^2 + bx^y \\ N &= (x + y)x^2 \end{aligned}$$

and let F be the vector field

$$F = (M, N)$$

so that

$$y' = -\frac{M}{N}.$$

Using the usual condition for exactness:

$$\begin{aligned} & D_2M - D_1N = 0 \\ \iff & 2xy + bx^2 = 3x^3 + 2xy \\ \iff & b = 3 \end{aligned}$$

so that

$$F(x, y) = xy^2 + 3x^2y.$$

It is easy to verify that F is conservative, with potential function $\phi(x, y) = \frac{x^2y^2}{2} + x^3y$. The level curves of ϕ provide solutions of (17):

$$\frac{x^2y^2}{2} + x^3y = c.$$

Math 4009 - Exercises

Philip Pennance[3] –Semester II de 2014-15

1. Let V be the vector field:

$$V(x, y) = (M(x, y), N(x, y))$$

where M, N, and the partial derivatives M_y, N_x are continuous on the entire plane.

   (a) What does it mean for V to be exact?

   (b) State a sufficient condition for V to be exact.

   (c) What does the exactness of V this tell you about solutions to the autonomous system:

$$\frac{dx}{dt} = -N(x, y)$$
$$\frac{dy}{dt} = M(x, y)$$

   (d) What does the exactness of V this tell you about solutions to the non-autonomous equation

$$y' = -\frac{M(x, y)}{N(x, y} \qquad (18)$$

   (e) Let $q(x)$ be a function of x only. Let $\hat{V}$ be the vector field:

$$\hat{V}(x, y) = (q(x)M(x, y), q(x)N(x, y))$$

   Show that if:

$$\frac{dq}{dx} = \frac{M_y - N_x}{N} q$$

   then $\hat{V}$ will be exact.

   (f) Apply these ideas to show that any solution to the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

   lies on a curve of the form

$$x^3 y + x^2 y^2/2 = c$$

2. For which constants k is

$$ke^x \cos y + (e^x \sin y - \sin y)y' = 0$$

an exact differential equation? In those cases, find the general solution of the differential equation.

3. Solve the equation.

$$x' = \frac{-x}{2t - xe^x}$$

4. Show that $\mu(x, y) = \dfrac{1}{xy^3}$ is an integrating factor of the equation

$$x^2 y^3 + x(1 + y^2)y' = 0$$

and find the solution in implicit form.

5. Find an integrating factor for the following equation and give the solution in implicit form:

$$3x^2 y + 2xy + y^3 + (x^2 + y^2)y' = 0.$$

6. Consider the family of curves

$$y^2 - x^2 = c$$

as c runs through the real numbers. Find another family of curves that is everywhere orthogonal to this one.

7. Along an adiabatic curve, the temperature θ of a homogeneous fluid body is a function $\hat{\theta}$ of volume v (i.e. $\theta = \hat{\theta}(v)$). The function $\hat{\theta}$ satisfies the differential equation

$$\hat{\theta}'(v) = -\frac{\Lambda(v, \hat{\theta}(v))}{K(v, \hat{\theta}(v))}$$

where Λ and K are constitutive functions. Find the function $\hat{\theta}$ for a fluid with

$$\Lambda(v, \theta) = \frac{1}{v^2}$$
$$K(v, \theta) = \frac{1}{3\theta + 1}$$

[3] https://pennance.us