Lecture Summary - Variation of Parameters

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1. Objective.

   We seek a particular solution of the second order differential equation

   $$x'' + p(t)x' + q(t)x = F(t) \tag{1}$$

   where p, q and F are continuous.

2. Claim. (Variation of parameters).

   If $\langle \phi_1, \phi_2 \rangle$ is a basis for the solution space of the second order homogeneous differential equation

   $$x'' + p(t)x' + q(t)x = 0, \tag{2}$$

   then there exist particular solution of (1) of the form

   $$x_p(t) = u_1(t)\phi_1(t) + u_2(t)\phi_2(t). \tag{3}$$

   Proof.

   (3) can be written as a scalar product

   $$x_p = \boldsymbol{u} \cdot \boldsymbol{\phi}. \tag{4}$$

   Substituting x_p in (1) and imposing the additional restriction

   $$\boldsymbol{u}' \cdot \boldsymbol{\phi} = 0 \tag{5}$$

   yields:

   $$\boldsymbol{u}' \cdot \boldsymbol{\phi}' + \boldsymbol{u} \cdot \boldsymbol{\phi}'' + p\boldsymbol{u} \cdot \boldsymbol{\phi}' + q\boldsymbol{u} \cdot \boldsymbol{\phi} = F.$$

   Since ϕ_1, ϕ_2 are solutions of the homogeneous equation

   $$\boldsymbol{u} \cdot \boldsymbol{\phi}'' + p\boldsymbol{u} \cdot \boldsymbol{\phi}' + q\boldsymbol{u} = 0.$$

   it follows that:

   $$\boldsymbol{u}' \cdot \boldsymbol{\phi}' = F. \tag{6}$$

   Equations (5) and (6) are equivalent to the system:

   $$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 &= 0 \\ u_1'\phi_1' + u_2'\phi_2' &= F. \end{aligned}$$

   By Cramer's rule,,

   $$u_1' = \frac{\begin{vmatrix} 0 & \phi_2 \\ F & \phi_2' \end{vmatrix}}{W}; \quad u_2' = \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & F \end{vmatrix}}{W} \tag{7}$$

   where

   $$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix}$$

   is the Wronkian. Since ϕ_1 and ϕ_2 are independent solutions of the differential equation, W is nowhere zero. Integrating (7) and substituting in (3) gives:

   $$\begin{aligned} x_p(t) &= \int^t \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{W(s)} F(s)ds \\ &= \int^t \frac{1}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} F(s)ds. \end{aligned}$$

   It is left as an exercise to verify that the lower limit of integration produces a term in the kernel and so any convenient value int the interval will do for the lower limit of integration.

3. Example.

   $$x'' - 5x' + 6x = 2e^t.$$

   Solution.

   The associated quadratic:

   $$\lambda^2 - 5\lambda + 6 = 0.$$

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has roots $\lambda = 2, 3$ and so a basis $\langle x_1, x_2\rangle$ for the kernel is given by

$$x_1(t) = e^{2t}$$
$$x_2(t) = e^{3t}$$

The Wronskian

$$W(s) = \begin{vmatrix} e^{2s} & e^{3s} \\ 2e^{2s} & 3e^{3s} \end{vmatrix} = e^{5s}.$$

Also

$$\begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix} = \begin{vmatrix} e^{2s} & e^{3s} \\ e^{2t} & e^{3t} \end{vmatrix} = e^{2s}e^{3t} - e^{2t}e^{3s}.$$

Hence a particular solution is

$$\begin{aligned} x_p(t) &= \int_0^t \frac{1}{W(s)} \begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix} 2e^s ds \\ &= \int_0^t \left[e^{3t-2s} - e^{2t-2s}\right] 2e^s ds \\ &= e^t. \end{aligned}$$

4. Example.

Destructive resonance of an undamped oscillator

$$x'' + \omega_0^2 x = F\cos\omega_0 t.$$

A basis for the solution of the homogeneous equation is given by

$$x_1(t) = \cos\omega t$$
$$x_2(t) = \sin\omega t.$$

The Wronskian:

$$W(s) = \begin{vmatrix} \cos\omega_0 s & \omega_0 \sin\omega_0 s \\ -\sin\omega_0 s & \omega_0\cos\omega_0 s \end{vmatrix} = \omega_0.$$

Hence a particular solution is

$$\begin{aligned} x_p(t) &= \int_0^t \frac{1}{W(s)} \begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix} F\cos\omega_0 t\, ds \\ &= \frac{Ft}{2\omega_0}\sin\omega_0 t + \frac{F}{4\omega_0^2}\cos\omega_0 t. \end{aligned}$$

Since the second term is in the kernel, it can be discarded, yielding the particular solution

$$x_p = \frac{Ft}{2\omega_0}\sin\omega_0 t.$$

5. Example. (Oscillator with small damping and arbitrary forcing term.)

$$x'' + 2\gamma x' + \omega_0^2 x = f(t).$$

If $0 < \gamma < \omega$ a basis for the solution of the homogeneous equation is given by

$$\phi_1(t) = e^{-\gamma t}\cos w_n t$$
$$\phi_2(t) = e^{-\gamma t}\sin w_n t$$

where $w_n = \sqrt{w_0^2 - \gamma^2}$. The Wronskian of ϕ_1 and ϕ_2 is

$$W(s) = w_n e^{-2\gamma s}$$

and

$$\begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} = e^{-\gamma s}e^{-\gamma t} \begin{vmatrix} \cos w_n s & \sin w_n s \\ \cos w_n t & \sin w_n t \end{vmatrix} = e^{-\gamma s}e^{-\gamma t}\sin w_n(t-s)$$

In this case, the variation of parameters formula yields the particular solution

$$\begin{aligned} x_p &= \int^t \frac{1}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} f(s) ds \\ &= \int_0^t \frac{1}{\beta} e^{-\gamma(t-s)}\sin w_n(t-s) f(s) ds. \end{aligned}$$

Remark. The above integral is a convolution

$$x_p = h * f.$$

where

$$h(t) = \begin{cases} \frac{1}{w_n} e^{-\gamma t} \sin w_n t & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

It follows that the transient solution $x_p(t)$ is the response at time t of a time invariant linear system with input signal f. The fact that $h(t) = 0$ for $t < 0$ is a causality condition. It guarantees that the system cannot respond to the future of the signal.