Lecture Summary - Oscillatory Systems

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1. Many damped oscillatory systems are described by differential equations of the form

$$x'' + 2\gamma x' + {\omega_0}^2 x = f(t) \qquad (1)$$

where $\gamma > 0$ is a damping parameter, $\omega_0 > 0$ is called the *undamped frequency* and $f(t)$ is an externally applied force. The parameter $\zeta = \dfrac{\gamma}{\omega_0}$ is called the *damping ratio.*

2. Reminder.

The general solution of (1) has the form

$$\phi(t) = x_n(t) + x_p(t)$$

where x is the general solution of the homogeneous equation and x_p is a particular solution of (1).

3. Homogeneous Equation.

A basis for the solution space of the homogeneous equation

$$x'' + 2\gamma x' + {\omega_0}^2 x = 0 \qquad (2)$$

is determined by the roots of auxiliary quadratic equation:

$$x^2 + 2\gamma x + {\omega_0}^2 = 0 \qquad (3)$$

There are three cases:

(a) Overdamping.
If $\zeta > 1$ then (3) has 2 real roots α_1, α_2 which satisfy

$$\alpha_1 \alpha_2 = {\omega_0}^2$$
$$\alpha_1 + \alpha_2 = -2\gamma$$

Since γ is positve, α_1, α_2 must be negative. Hence the general solution

$$x_n(t) = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$$

decays exponentially with time.

(b) Critical Damping.
If $\zeta = 1$ then (3) has equal roots $\alpha_1 = \alpha_2 = -\gamma$ yielding a general solution

$$x_n(t) = (c_1 + c_2 t)e^{-\gamma t}.$$

(c) Small Damping.
If $\zeta < 1$ then (3) has complex roots $-\gamma \pm i\omega_n$ where

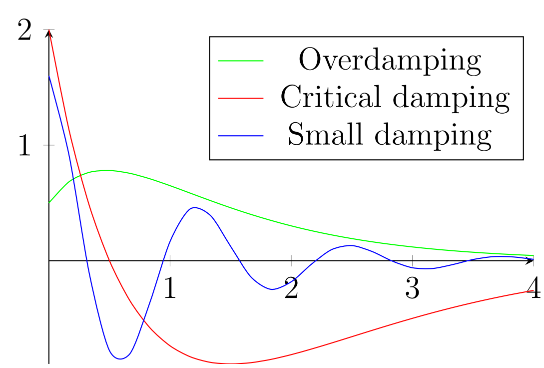
$$\omega_n = \sqrt{\omega_0^2 - \gamma^2}.$$

In this case the general solution is

$$x_n(t) = e^{-\gamma t}(c_1 \cos \omega_n t + c_2 \sin \omega_n t).$$

The quantity ω_n, called the *natural frequency,* decreases as the damping increases. It is left as an exercise to show that there exist constants C, δ such that

$$x_n(t) = Ce^{-\gamma t}(\cos(\omega_n t - \delta).$$



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In all cases, the general solution of the homogeneous equation (2) is transient, decaying exponentially with time. Thus at large times the particular solution dominates.

4. Particular Solution.

   For an external force of the form

$$f(t) = F \cos \omega t,$$

   a particular solution of (1) can be obtained by noting that if

$$z_p(t) = x_p(t) + i y_p(t)$$

   is a solution of

$$z'' + 2\gamma z' + {\omega_0}^2 z = F e^{i\omega t} \qquad (4)$$

   then the real part $x_p(t)$ will be a solution of

$$x'' + 2\gamma x' + {\omega_0}^2 x = F \cos \omega t \qquad (5)$$

5. Claim.

   Equation (4) has a particular solution of the form

$$z_p(t) = A e^{i\omega t}.$$

   Proof.

   Substitution of z_p in (4) yields

$$\begin{aligned} A &= \frac{F}{({\omega_0}^2 - \omega^2) - 2\gamma\omega i} \\ &= \frac{({\omega_0}^2 - \omega^2 + 2\gamma\omega i)F}{({\omega_0}^2 - \omega^2)^2 + 4\gamma^2\omega^2} \end{aligned}$$

   In polar form of $A = |A|e^{i\phi}$ where

$$\tan \phi = \frac{2\gamma\omega}{{\omega_0}^2 - \omega^2}. \qquad (6)$$

   and

$$|A| = \frac{F}{\sqrt{({\omega_0}^2 - \omega^2)^2) + 4\gamma^2\omega^2}}. \qquad (7)$$

   The solution for input $F \cos \omega t$ is

$$\begin{aligned} x_p &= \mathrm{Re}\, z_p \\ &= \mathrm{Re}\, A e^{i\omega t} \\ &= \mathrm{Re}\, |A| e^{i\omega t + \phi} \\ &= |A| \cos(\omega t + \phi). \end{aligned}$$

   To see how the amplitude varies with the applied frequency notice that the expression under the square root in equation (7) is

$$\begin{aligned} &(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 \\ &= \omega^4 - (2\omega_0^2 - 4\gamma^2)\omega^2 + \omega_0^4 \\ &= [\omega^2 - (\omega_0^2 - 2\gamma^2)]^2 - [\omega_0^2 - 2\gamma^2]^2 + \omega_0^4 \\ &= [\omega^2 - (\omega_0^2 - 2\gamma^2)]^2 - 4\gamma^4 + 2\gamma^2\omega^2 \\ &= [\omega^2 - (\omega_0^2 - 2\gamma^2)]^2 + 2\gamma^2[\omega_0^2 - \gamma^2] \\ &= [\omega^2 - (\omega_r^2)]^2 + 2\gamma^2\omega_n^2 \end{aligned}$$

   where

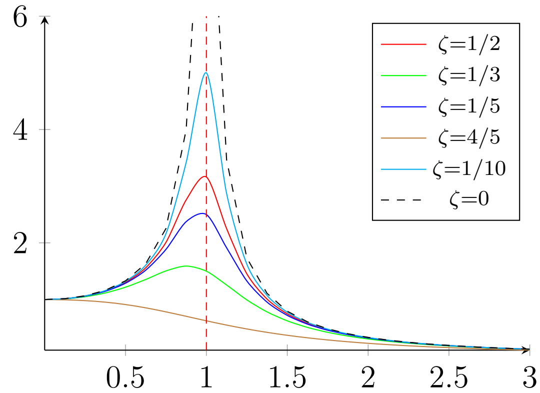
$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

   is called the *resonance frequency.*

6. Corollary.

   (a) If $0 < \zeta < 1/\sqrt{2}$, then $\omega_0^2 - 2\gamma^2 > 0$ and the amplitude has a maximum when $\omega = \omega_r$, the resonant frequency.

   (b) If $1/\sqrt{2} \le \zeta \le 1$, then $\omega_0^2 - 2\gamma^2 \le 0$ and the amplitude of the cosine is a decreasing function of ω with a maximum amplitude for a static applied force ($\omega = 0$).



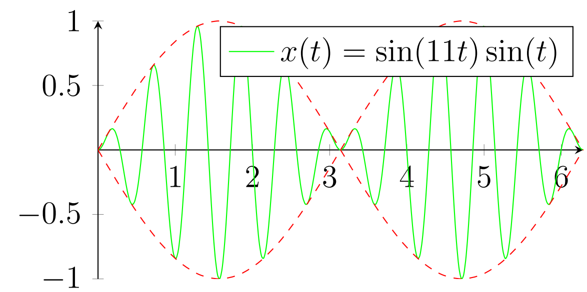
Amplitude of steady state solution as a function of ω for different values of ζ.

7. Remark.

   The undamped frequency ω_0, the natural frequency ω_n and the resonance frequency ω_r are connected by the equation
   $$\omega_n{}^2 = \frac{\omega_0^2 + \omega_r^2}{2}.$$

8. The Case of Zero Damping.

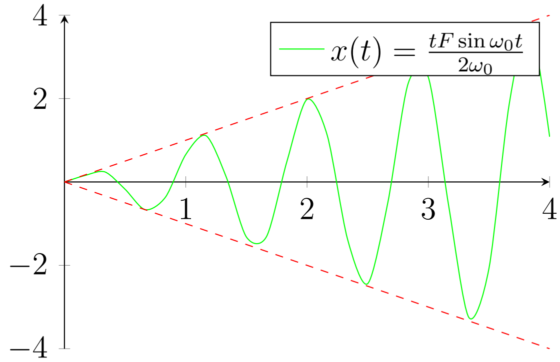
   If $\gamma = 0$ the homogeneous equation
   $$x'' + \omega_0{}^2 x = 0 \tag{8}$$
   has general solution
   $$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$
   If $\omega \neq \omega_0$ the non homogeneous equation
   $$x'' + \omega_0{}^2 x = F \cos \omega t \tag{9}$$
   Putting $\gamma = 0$ in (6) and (7) yields a particular solution
   $$x_p(t) = A \cos \omega t$$
   where
   $$A = \frac{F}{|\omega_0^2 - \omega^2|} \tag{10}$$
   and so the general solution of (9) is
   $$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + A \cos \omega t.$$
   If $x(0) = x'(0) = 0$, then $c_1 = -A$ and $c_2 = 0$, and so
   $$x(t) = A\left(\cos \omega t - \cos \omega_0 t\right).$$
   Using standard trigomometrical identities, this can be written
   $$x(t) = 2A \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$
   If $\omega_0 - \omega$ is significantly smaller than $\omega_0 + \omega$, the solution will be a sinusoidal oscillation of angular frequency $\frac{\omega_0+\omega}{2}$ with amplitude modulated by the lower angular frequency $\frac{\omega_0-\omega}{2}$, a phenomena called *beats*. The diagram below shows the case with $\omega_0 = 12$ and $\omega = 10$.



Beats: $\omega_0 = 12$, $\omega = 10$

9. Destructive Resonance.

   Notice $|x(t)| \to \infty$ as $\omega \to \omega_0$ a phenomena called *destructive resonance*. If $\omega = \omega_0$, the function $t \mapsto \cos \omega_0 t$ lies in the kernel of the operator $D^2 + \omega_0^2$ and so no multiple of this can be particular solution. However, there does exist a particular solution of the form
   $$x_p(t) = t(A \cos \omega_0 t + B \sin \omega_0 t).$$
   Substitution in (9) yields $A = 0$ and $B = \dfrac{F}{2\omega_0}$. Hence, in the case of destructive resonance, the general solution of (9) is
   $$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{tF \sin \omega_0 t}{2\omega_0}.$$



Destructive Resonance