Ordinary Differential Equations - Introduction

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1. Let $D \subseteq \mathbb{R}^2$ and $F : D \to \mathbb{R}$. The expression:

$$x' = F(t, x), \quad (t, x) \in D \qquad (1)$$

is called the *first order ordinary differential equation determined by F.*

If $(t_0, x_0) \in D$, the pair:

$$\begin{cases} x' = F(t, x), & (t, x) \in D \\ x(t_0) = x_0. \end{cases} \qquad (2)$$

is called the *initial value problem* (IVP) determined by the differential equation (1) and the *initial condition* $x(t_0) = x_0$.

2. Let I be an interval. A function

$$\phi : I \to \mathbb{R}$$

is a *solution* of (1) if

$$\phi'(t) = F(t, \phi(t))$$

for all $t \in I$. Moreover, ϕ is said to be a *solution of the initial value problem* (2) if it also satisfies $\phi(t_0) = x_0$.

3. Example. Let

$$D = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}.$$

Then (exercise)

$$\phi(t) = -\sqrt{1 - t^2}$$

is a solution of the initial value problem

$$\begin{cases} x' = -\dfrac{t}{x}, & (t, x) \in D \\ x(0) = -1. \end{cases}$$

4. Properties of Solutions:

Let $\phi : I \to \mathbb{R}$ be a solution of (1). Then (exercise):

   (a) ϕ is differentiable on I.

   (b) If

$$G_\phi = \{(t, \phi(t) : t \in I\}$$

   is the graph of ϕ, then $G_\phi \subseteq D$.

   (c) If $J \subset I$ is an interval, and ψ is the restriction of ϕ to J, then ψ is also a solution of (1)

5. Let $\phi : \bar{I} \to \mathbb{R}$ be a function. A function $\bar{\phi} : \bar{I} \to \mathbb{R}$ is an *extension* of ϕ if:

   (a) $I \subseteq \bar{I}$

   (b) $G_\phi \subseteq G_{\bar{\phi}}$

6. Let I be an interval. A function $\phi : \bar{I} \to \mathbb{R}$ is a *maximal* solution of (1) if:

   (a) ϕ is a solution of (1).

   (b) ϕ has no extension to a solution $\phi : \bar{I} \to \mathbb{R}$ where $\bar{I}$ properly contains I.

7. A maximal solution of (2) is not be necessarily unique. For example:

   (a) $\phi(t) = \begin{cases} 0 & \text{if } t < 4, \\ (t-4)^2 & \text{if } t \geq 4 \end{cases}$

   (b) $\phi(t) = \begin{cases} 0 & \text{if } t < 3, \\ (t-3)^2 & \text{if } t \geq 3 \end{cases}$

   (c) $\phi(t) = 0, \quad t \in \mathbb{R}$

are all maximal solutions of the initial value problem:

$$\begin{cases} x' = 2\sqrt{x} \\ x(3) = 0 \end{cases}$$

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8. Example.

Show that the IVP

$$\begin{cases} x' = 0, \quad (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{3}$$

has the unique solution

$$\phi(t) = x_0 \tag{4}$$

Solution.

Clearly (7) is a solution. Let I be an interval and $\psi(t), t \in I$ be any other solution. Then,

$$\begin{cases} \psi'(t) = 0, \quad (t,x) \in \mathbb{R}^2 \\ \psi(t_0) = x_0. \end{cases} \tag{5}$$

It follows that

$$\psi'(t) - \phi'(t) = 0, \quad t \in I$$

Since I is an interval, it follows from the mean value theorem that $\psi - \phi$ is constant on I. Hence, for some $c \in \mathbb{R}$

$$\psi(t) = \phi(t) + c, \quad t \in I.$$

Putting $t = t_0$ and using the initial condition shows that $c = 0$. Hence

$$\psi(t) = \phi(t), \quad t \in I.$$

This means that ψ is a restriction of ϕ to the interval I and so ϕ is the unique solution with maximal domain.

9. Claim.

Let $q(t)$ be continuous on an interval $[t_0, t_1]$. Then the IVP

$$\begin{cases} x' = q(t), \quad (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{6}$$

has the unique solution

$$\phi(t) = x_0 + \int_{t_0}^{t} q(\tau)\, d\tau. \tag{7}$$

Any other solution is a restriction of ϕ.

Proof.

It follows, from the Fundamental Theorem that ϕ is a solution. Let I be an interval and $\psi(t), t \in I$ be any other solution. Then,

$$\begin{cases} \psi'(t) = q(t), \quad t \in I \\ \psi(t_0) = x_0. \end{cases} \tag{8}$$

It follows that

$$\psi'(t) - \phi'(t) = 0, \quad t \in I$$

Since I is an interval, it follows from the mean value theorem that $\psi - \phi$ is constant on I. Hence, for some $c \in \mathbb{R}$

$$\psi(t) = \phi(t) + c, \quad t \in I.$$

Putting $t = t_0$ and using the initial condition shows that $c = 0$. Hence

$$\psi(t) = \phi(t), \quad t \in I.$$

This means that ψ is merely a restriction of ϕ to the interval I. and so ϕ is the unique solution with maximal domain.

10. Example.

Solve the IVP

$$\begin{cases} x' = t^2 \\ x(t_0) = x_0. \end{cases} \tag{9}$$

Solution.

$$\begin{aligned} \phi(t) &= x_0 + \int_{t_0}^{t} \tau^2\, d\tau \\ &= x_0 + \frac{t^3}{3} - \frac{t_0^3}{3}. \end{aligned}$$