Math 4009 - Homework: Simple Harmonic Motion

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1. In the interior of a homogeneous spherical planet, the gravitational force on a mass m, located at distance x from the centre, is $F(x) = -kmx$ where k is a constitutive constant. A tunnel is constructed from pole to pole and a ball of mass m is placed in the tunnel. Assume that there is no air resistance or friction or other force within the tunnel.

   a) Using Newton's law, write an initial value problem (IVP) to determine the motion when the ball is released from rest at the center of the earth ($x = 0$) at time $t = t_0$. Show that the unique solution of this IVP is the zero function on $[t_0, \infty)$.

   b) Find the position of the ball as a function of time when dropped from a pole at time $t = 0$ and prove that the solution is unique.

   c) In (b) find the velocity of the ball when $x = 0$ and the time to go from pole to center.

   d) In (b), show that the total energy is a constant of the motion.

   e) A straight tunnel is drilled through the earth from London to Moscow and another from London to New York. A tube is dropped in the tunnel to Moscow at time $t = 0$ and travels under the influence only of gravity towards Moscow. Another tube is dropped simultaneously in the tunnel to New York. Assuming as before that there is no air resistance or friction or other force (apart from gravity) within the tunnels show that the tubes arrive at their respective destinations and moreover at the same time.

2. At time t, a simple pendulum length l makes a angle $\theta(t)$ with the vertical. In the case of small oscillations in a gravitational field of constant g, this angle satisfies the differential equation $l\theta''(t) = -g\theta$

   a) Find a constant c such that, after the change of time scale $\hat{\theta} = \theta(ct)$, the function $\hat{\theta}$ satisfies: $\hat{\theta}'' + \hat{\theta} = 0$.

   b) Use the results of part (a) to find the unique solution of the equation of a pendulum which satisfies the initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = 0$. Show that the solution is periodic with period $T = 2\pi\sqrt{l/g}$.

[1] http://pennance.us