

Math 4009 - Homework on the Wronskian

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1. Are the following pairs of functions linearly independent on the specified interval:

a) $f(x) = e^{3x}$, $g(x) = e^{3(x-1)}$, $x \in \mathbb{R}$.

b) $f(t) = t$, $g(t) = t^{-1}$, $t > 0$

2. Suppose that the Wronskian of two functions y_1, y_2 is

$$W(y_1, y_2; t) = t \sin^2 t, \quad t \in \mathbb{R}$$

Are the two functions linearly independent?

3. Prove, that if two functions α and β have a maxima at the same point of an interval I , then they cannot be a linearly independent set of solutions of an equation

$$x'' + p(t)x' + q(t)x = 0$$

where p and q are continuous on I .

4. Prove or disprove: If the Wronskian of f and g is identically equal to zero on an interval I , then f and g are linearly dependent.

5. Consider the equation $y'' + \cos(x)y' + \sin(x)y = 0$

a) Let ϕ be a nontrivial solution, and let $\psi(x) = \phi(x + 2\pi)$. Prove that $\psi(x)$ is also a solution.

b) Show that $\phi(x)$ is a periodic solution of period 2π if, and only if, $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$.

c) Let $\phi_1(x), \phi_2(x)$ be two solutions satisfying $\phi_1(0) = 1$, $\phi_1'(0) = 0$, $\phi_2(0) = 0$, $\phi_2'(0) = 1$. Show that there are constants a and b such that:

$$\phi_1(x + 2\pi) = a_1\phi_1(x) + b_2\phi_2(x)$$

6. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation

$$y'' + p(t)y' + q(t)y = 0$$

with continuous coefficients?

7. Find all solutions of the differential equation

$$ty'' - (1 + t)y' + y = 0, \quad t > 0$$

on the given that $y_1 = 1 + t$ is a solution.

Hint: Suppose y_2 is another solution, such that y_1 and y_2 are linearly independent. Find the possible functions $W(y_1, y_2)$. Use this to find one such y_2 . Find the Wronskian, using Abel's theorem, and from this construct a differential equation satisfied by y_2 .

8. Are the functions $f(x) = e^{3x}$, $g(x) = e^{3(x-1)}$ linearly independent? Prove your
9. Suppose ϕ and ψ are two functions on $I = (0, \infty)$ such that the Wronskian $W(t) = t - 1$. Can ϕ and ψ be independent solutions of a differential equation of the form $x'' + p(t)x' + q(t)x = 0$ where p and q are continuous on I ? Justify.

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