Math 4009 - Homework: First Order Linear ODE's

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1. Show that $L : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}]$ given by

$$L(\phi) = \phi' + (\sin t)\phi$$

is a linear map.

   (a) Find the general solution of the homogeneous equation

$$x' + (\sin t)x = 0$$

   (b) Find the general solution of

$$x' + (\sin t)x = \sin t$$

   (c) Find the unique solution satisfying the initial condition $x(0) = 1$.

2. Solve:

   (a) $x' = x - t$

   (b) $x' = x + e^t$

   (c) $x' = -x \cot t + 5e^{\cos t}$

   (d) $y' + y/t = 3 \cos 2t$, $t > 0$

3. Solve the linear first order IVP:

$$x' + p(t)x = 2; \quad x(1) = 5$$

where

$$p(t) = \begin{cases} 2 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Hint: Solve the IVP with $p(t) = 2$ on the interval $[0, 1]$ and use $x(1)$ as new initial value on the interval $[1, \infty)$ with $p(t) = 0$.

4. Solve the IVP:

$$(1 - t^2)x' = tx + t, \qquad x(0) = x_0.$$

Draw a direction field and sketch the solutions for $x_0 \in \{-2, -1, 0, 1, 2\}$.

5. Compute the value of y_0 for which the solution of the initial value problem:

$$y' - y = 1 + 3 \sin x, \quad y(0) = y_0$$

remains finite (as $x \to \infty$). Use a program to draw direction fields and typical solutions:

   (a) for $(x, y) \in [0, 4\pi] \times [-6, 6]$.

   (b) for $(x, y) \in [-20, 20] \times [-6, 6]$.

6. Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t; \quad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

7. (a) Let S be the set of solutions of $x' + p(t)x = q(t)$ where p, q are continuous on an interval I. Let $\hat{x} \in S$ and let K be the set of solutions of the "homogeneous equation" $x' + p(t)x = 0$. Show that $S = \{\hat{x} + \phi \mid \phi \in K\}$.

   (b) Use the observation in part (a) to find the general solution of the ordinary differential equation $x' = x - 3$

8. Find the solution of the initial value problem:

$$(1 - t^2)x' = tx + t, \qquad x(0) = x_0.$$

Find the maximal interval $I = (a, b)$ on which the solution of the initial value problem exists.

9. Compute the value of x_0 for which the solution of the initial value problem:

$$x' = x + \cos t, \quad x(0) = x_0$$

remains finite

[1] https://pennance.us