Math 4009 - Linear Equations

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1. Show that $L : \mathbb{R} \to \mathbb{R}$ given by
$$L(x) = -x - 1$$
is NOT linear.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be linear. Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be linear. Show that there exist constants a, b, c, d such that $f(x, y) = (ax + by, cx + dy)$ for all $x, y \in \mathbb{R}$.

4. Let $\mathcal{C}^1[\mathbb{R}]$ be the set of functions with continuous derivative over $\mathbb{R}$ and $\mathcal{C}^0[\mathbb{R}]$ the set of continuous functions over $\mathbb{R}$.
   (a) Show that $\mathcal{C}^0[\mathbb{R}]$ is a real vector space under the usual operations of addition of functions and multiplication of a function by a real number.
   (b) Show that $\mathcal{C}^1[\mathbb{R}]$ is a subspace of $\mathcal{C}^0[\mathbb{R}]$.
   (c) Give an example of a function belonging to $\mathcal{C}^0[\mathbb{R}]$ but not $\mathcal{C}^1[\mathbb{R}]$.

5. Which of the following equations are linear? Prove your answers.
   (a) $x' = 2\sqrt{x}$.
   (b) $x'' = 1 - \dot{x}^2$
   (c) $x'' + x = 0$
   (d) $xx' = 0$

6. Which of the following functions from $\mathcal{C}^1[\mathbb{R}]$ to $\mathcal{C}^0[\mathbb{R}]$ are linear? Prove your answers.
   (a) $T(x) = x'' + x + 1$
   (b) $T(x) = xx'$
   (c) $T(x) = xx' + 1$

7. Let $L : \mathcal{C}^2[\mathbb{R}] \to \mathcal{C}^0[\mathbb{R}]$ be given by $L(x) = x'' + bx' + cx$ where a, b, c are constants. Show that L is linear.

8. Show that a subset of a linearly independent set is linearly independent.

9. Suppose that $\phi(t)$ is a solution of
$$x'' + p(t)x' + q(t)x = g(t),$$
where g is not the zero function. For which values of c is $c\phi$ a solution?

10. Suppose that the Wronskian of two functions f and g is is not equal to the zero function on an interval I. Show that f and g are linearly independent. Find a counter example to the converse.

11. Let $p, q : [t_0, t_1] \to \mathbb{R}$ be continuous.
    (a) Show that if ϕ is a solution of the linear differential equation:
    $$x' + p(t)x = q(t), \quad (t, x) \in [t_0, t_1] \times \mathbb{R}$$
    and P is any antiderivative of p then $\dfrac{d}{dt}\left[\phi(t)e^{P(t)}\right] = q(t)e^{P(t)}$.
    (b) Let p be continuous on the interval $[t_0, t_1]$. Find the kernel of the linear map
    $$L : \mathcal{C}^1[t_0, t_1] \to \mathcal{C}^0[t_0, t_1]$$
    given by
    $$L(x) = x' + p(t)x$$
    (c) Use the observation in part (a) to find the general solution of the ordinary differential equation $x' = x - t$

12. Consider the differential equation
$$x' = 2x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$
Let S be the set of all solutions.

[1] https://pennance.us

(a) Show that the map $L : \mathcal{C}^1 \to \mathcal{C}^0$ given by

$$L(x) = x' - 2x$$

is linear.

(b) Show that S is a vector space under addition.

(c) Show that the *initial value* map $E : S \to \mathbb{R}$ given by $E(\phi) = \phi(0)$ is linear, injective and surjective. What does this say about the dimension of S?.

13. (a) Define linearly independent list of vectors.

(b) Show that $(1,0),(0,2)$ is a linearly independent list of vectors in $\mathbb{R}^2$

(c) Show that $(1,0),(0,2)$ is a spanning set in $\mathbb{R}^2$

(d) Show that a sublist of a linearly independent list is also linearly independent.

(e) Show that $\langle \sin, \cos \rangle$ is a linearly independendent list of vectors in the space of continuous functions $\mathcal{C}^0[\mathbb{R}]$.

14. Are the functions $\alpha(t) = t$, $\beta(t) = t^{-1}$ linearly independent on the interval $t > 0$? Prove your answer.

15. Let $\{v, w, u\}$ be a linearly independent set. Show carefully that $\{v, w\}$ is also linearly independent.

16. Let V, W be vector spaces, $L : V \to W$ be a linear map and $b \in \operatorname{Im} L$. Let $u, v \in V$ be two distinct solutions of the equation $Lx = b$.

(a) Show that for all $t \in \mathbb{R}$,

$$x = tu + (1-t)v$$

is also a solution of $Lx = b$.

(b) Show that every solution of $L(x) = b$ is of the form $u + \alpha$ where α is an element of the kernel of L and u is any particular solution.