Math 4009 - Autonomous Differential Equations

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1. Let U be an open interval and $v : U \to \mathbb{R}$ continuous. Let x_0 be a singular point of v Show that $\phi(t) = x_0$ is a solution of the initial value problem $x' = v(x), \quad x(t_0) = x_0$

2. *a*) Find an infinite number of solutions of the initial value problem

$$x' = 2\sqrt{x}, \quad x(0) = 0$$

   b) Find **all** maximal solutions of the differential equation $x' = 2\sqrt{x}, \quad x \geq 0$.

3. Consider the ordinary differential equation $x' = 2\sqrt{x}$.

   a) Let $c > 0$. Show that if ϕ is a solution on the interval $t > 0$ then so is $\psi(t) = \phi(t - c), \quad t > c$

   b) Find three distinct solutions of the initial value problem

$$x' = 2\sqrt{x}, \quad x(4) = 0$$

   which are defined on all of $\mathbb{R}$.

   c) Draw the direction field and solutions of $x' = 2\sqrt{x}$.

4. *a*) Let $v(x) = x^2, \quad x \in \mathbb{R}$. Discuss the nature of the singular point of v.

   b) Find **all** solutions of the differential equation $x' = x^2, \quad x \in \mathbb{R}$.

   c) Sketch the solutions $x' = x^2$.

5. Consider the equation

$$x' = 4(1 - x)^2, \qquad k > 0 \tag{1}$$

   a) Find the equilibrium solution(s) of the equation.

   b) Let ϕ be a solution of (1). Show that ϕ is strictly increasing for $x < 1$ and also for $x > 1$.

   c) Solve equation (1) subject to the initial condition $x(0) = x_0$. Using your result in (b) or (c) draw a qualitative picture of the solutions.

6. Arrange the following equations in qualitatively equivalent classes.

   a) $x' = \cosh x$.

   b) $x' = (x - a)^2$.

   c) $x' = \sin x$.

   d) $x' = \cos x - 1$.

   e) $x' = \cosh x - 1$.

   f) $x' = \sin 2x$.

   g) $x' = e^x$.

   h) $x' = \sinh^2(x - b)x$.

7. Find all solutions of $x' = \sqrt{1 - x^2}$.

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8. For each of the following autonomous ODE's, find equilibrium values of the function and make a qualitative sketch of typical solution curves. Label parts of the y axis where $y' > 0$ or $y' < 0$.

   $a)$ $dy/dt = 3 - y$

   $b)$ $dy/dt = y(y-2)^2$

9. Consider the autonomous equation $x' = v(x)$ in one dimension. Discuss the nature (attractor, repellor, shunt) of the possible singular point of v.