

Math 4009 - Introduction

Philip Pennance¹

1. Let $D \subseteq \mathbb{R}^2$ and $F : D \rightarrow \mathbb{R}$ a real valued function. By the *differential equation determined by F* we mean the expression

$$x' = F(t, x), \quad (t, x) \in D \quad (1)$$

- (a) Define carefully the term *solution* of (1)
- (b) State the fundamental theorem of calculus.
- (c) Let $q : [t_0, t_1] \rightarrow \mathbb{R}$ be continuous. Show that $\phi(t) = x_0 + \int_{t_0}^t q(\tau) d\tau$ is a solution of the initial value problem

$$x' = q(t), \quad x(t_0) = x_0 \quad (2)$$

- (d) Show that any other solution of (2) is a restriction of ϕ .
 - (e) Find the unique solution of the the second order IVP $x'' = -1$ with initial conditions $x(t_0) = x_0, x'(t_0) = v_0$.
2. A population of bacteria doubles every hour. Let $x(t)$ be the population at time t . Let $x_0 = x(t_0)$ be the initial population at some time t_0 .

- (a) Show, by induction, that the population n hours later is given by:

$$x(t_0 + n) = x_0 2^n, \quad n \in \mathbb{N}$$

- (b) Assuming that the size of the above population can be modelled by a continuous variable find the constant k such that $x(t) = x(t_0)e^{k(t-t_0)}$.
- (c) Consider the initial value problem:

$$x' = kx, \quad x(t_0) = x_0$$

- i. Verify that $\phi(t) = x_0 e^{k(t-t_0)}$ is a solution.
 - ii. Show that any other solution $\psi : I \rightarrow \mathbb{R}$, where I is an interval, is a restriction of ϕ . Hint: Consider the derivative of the function $\psi(t)e^{-k(t-t_0)}$.
3. Find maximal solutions of the following initial value problems:

(a) $x'' = -g, \quad x(t_0) = x_0, \quad x'(t_0) = v_0.$

(b) $x' - (\sin 2t)x = 0, \quad x(0) = 0$

(c) $y' = -\frac{x}{y}, \quad y(0) = \pi$

4. Evaluate $D_2 f$ at the point $(2, 3)$ if $f(x, y) = \frac{\sin(x-2)}{1 + \sin(y^2 + \tan xy)}$

¹<https://pennance.us>

5. A function $F : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is called *separable on D* if there exist functions f, g such that $F(x, y) = f(x)g(y)$ for all $(x, y) \in D$.

(a) Let $h(x, y) = xy - xy^3 + x^2y - x^2y^3$. Show that h separable on $\mathbb{R}^2$.

(b) Let $F : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be separable. Suppose that the partial derivatives D_1F , D_2F , D_1D_2F exist on D . Show that $F D_1 D_2 F = D_1 F D_2 F$.

6. The rate of decay of a radioactive substance is proportional to the amount present. Suppose that one third decays after five years. What proportion is left after 10 years? Prove your answer. What is the half life of the substance?

7. The proportion $P(t)$ of people suffering from a deadly flu virus at time t is given by

$$P' = kP(1 - P)$$

where k is a constitutive constant. Find $P(t)$ subject to the initial condition $P(0) = a$. Sketch a graph of the solution.

8. The velocity $v(t)$ of a parachutist whose chute fails to open is given by the initial value problem: $v' = a - bv^2$, $v(0) = 0$ where a and b are constitutive constants. Find $v(t)$ and sketch the graph of v for $t > 0$.
9. Consider the family of curves $y^2 - x^2 = c$, as c runs through the real numbers. Find another family of curves that is everywhere orthogonal to this one. (In other words, at almost every point in the plane, the tangent to the corresponding curve in the first family is perpendicular to the tangent to the corresponding curve in the second family.)