

Takehome Exam 3 — Solutions

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1. Show that if $y = \phi(t)$ is a solution of

$$y'' + p(t)y' + q(t)y = g(t),$$

where g is not identically zero, then $y = c\phi(t)$ is *not* a solution for any constant $c \neq 1$.

Solution. Let $L(y) = y'' + p(t)y' + q(t)y$. Then L is linear and $L(\phi) = g$. Hence

$$L(c\phi) = cL(\phi) = cg.$$

If $c\phi$ were a solution of the inhomogeneous equation we would have $cg = g$, so $(c-1)g \equiv 0$. Since $g \not\equiv 0$, it follows that $c = 1$.

2. Use reduction of order to find the general solution of

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0,$$

given that $y(t) = t$ is a solution.

Solution. Rewrite the equation in standard form

$$y'' + p(t)y' + q(t)y = 0, \quad p(t) = -1 - \frac{2}{t}.$$

Since $\phi_1(t) = t$ is a solution, a second independent solution is

$$\phi_2(t) = \phi_1(t) \int \frac{e^{-P(t)}}{\phi_1(t)^2} dt,$$

where P is any antiderivative of p . Taking

$$P(t) = \int \left(-1 - \frac{2}{t}\right) dt = -t - 2 \ln t$$

gives

$$\phi_2(t) = t \int \frac{e^{t+2 \ln t}}{t^2} dt = t \int \frac{e^t \cdot t^2}{t^2} dt = t \int e^t dt = te^t.$$

Thus $\{t, te^t\}$ is a fundamental set on $(0, \infty)$, and the general solution is

$$y(t) = c_1t + c_2te^t.$$

3. (a) Let

$$h(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Compute the convolutions $h * \delta$ and $h * h$.

(b) Find the function $h(t)$ such that the solution of

$$y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

is given by the convolution $h * f$.

Solution.

(a) By the sifting property,

$$(h * \delta)(t) = \int_{\mathbb{R}} h(\tau) \delta(t - \tau) d\tau = h(t).$$

Algebraically, δ is the identity for the convolution product.

Next,

$$(h * h)(t) = \int_{\mathbb{R}} h(\tau) h(t - \tau) d\tau.$$

The integrand is 1 precisely when $\tau \in [0, 1]$ and $t - \tau \in [0, 1]$, i.e. when τ lies in $[0, 1] \cap [t - 1, t]$. Hence

$$(h * h)(t) = \begin{cases} 0 & t < 0, \\ t & 0 \leq t \leq 1, \\ 2 - t & 1 \leq t \leq 2, \\ 0 & t \geq 2. \end{cases}$$

This is a triangular pulse supported on $[0, 2]$.

(b) The kernel h is the causal impulse response:

$$\begin{cases} h'' + 2h' + 2h = \delta, \\ h(0^-) = 0, \quad h'(0^-) = 0. \end{cases}$$

Let $\hat{h}(s) = \mathcal{L}\{h\}(s)$. Then

$$s^2 \hat{h}(s) + 2s \hat{h}(s) + 2\hat{h}(s) = 1, \quad \hat{h}(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1)^2 + 1}.$$

Inverting the transform yields

$$h(t) = e^{-t} \sin t \cdot \mathbf{1}_{[0, \infty)}(t).$$

The solution of the initial-value problem is therefore

$$y(t) = (h * f)(t) = \int_0^t e^{-(t-\tau)} \sin(t - \tau) f(\tau) d\tau \quad (t \geq 0).$$

4. Let $b, c \in \mathbb{R}$ and consider

$$x'' + bx' + cx = 0. \quad (1)$$

Let α_1, α_2 be the roots of $x^2 + bx + c = 0$.

- (a) Show that if $\alpha_1 = \alpha + i\beta$ with $\beta \neq 0$, then $\alpha_2 = \alpha - i\beta$.
- (b) Show that $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$ are linearly independent solutions of (??).

Solution.

- (a) The coefficients b, c are real, so nonreal roots of the characteristic polynomial occur in conjugate pairs. Thus $\alpha_2 = \overline{\alpha_1} = \alpha - i\beta$.
- (b) Let $z = \alpha + i\beta$. Then $z^2 + bz + c = 0$, and

$$(e^{zt})'' + b(e^{zt})' + ce^{zt} = e^{zt}(z^2 + bz + c) = 0.$$

Hence e^{zt} is a complex solution. Writing

$$e^{zt} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

and taking real and imaginary parts (which is legitimate because the equation has real coefficients) shows that

$$e^{\alpha t} \cos \beta t = \Re(e^{zt}), \quad e^{\alpha t} \sin \beta t = \Im(e^{zt})$$

are real solutions.

Linear independence follows from the Wronskian

$$W(t) = e^{2\alpha t} \beta.$$

Since $\beta \neq 0$, one has $W \neq 0$.

5. The current $i = i(t)$ in a simple RLC circuit satisfies

$$Li'' + Ri' + \frac{1}{C}i = 0,$$

where $C > 0$, $R \geq 0$, and $L > 0$.

- (a) Show that every solution $i(t)$ is bounded on $[0, \infty)$.
- (b) Show that if $R > 0$, then $i(t) \rightarrow 0$ as $t \rightarrow \infty$.
- (c) Under what condition is the solution oscillatory?

Solution. Set

$$\gamma = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{\gamma}{\omega_0} = \frac{R}{2} \sqrt{\frac{C}{L}}.$$

The equation becomes

$$i'' + 2\gamma i' + \omega_0^2 i = 0.$$

The form of the general solution depends on the discriminant $\gamma^2 - \omega_0^2$, or equivalently on ζ :

- $\zeta > 1$ (overdamped): $i(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$ with $r_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} < 0$ whenever $R > 0$;
- $\zeta = 1$ (critically damped): $i(t) = (c_1 + c_2 t) e^{-\gamma t}$;
- $\zeta < 1$ (underdamped): $i(t) = e^{-\gamma t} (c_1 \cos \omega t + c_2 \sin \omega t)$, $\omega = \omega_0 \sqrt{1 - \zeta^2}$.

In every case the solutions are bounded on $[0, \infty)$ (if $R = 0$ then $\gamma = 0$ and one obtains simple harmonic motion, which is bounded). If $R > 0$ then $\gamma > 0$, and the factor $e^{-\gamma t}$ forces $i(t) \rightarrow 0$ as $t \rightarrow \infty$.

The solutions are oscillatory precisely when the characteristic roots are nonreal, i.e. when $\zeta < 1$, or equivalently

$$\frac{R}{2} \sqrt{\frac{C}{L}} < 1.$$

(See the [class notes](#) for the detailed case analysis.)

6. Consider

$$y'' + \cos(x) y' + \sin(x) y = 0.$$

- Let ϕ be a solution and set $\psi(x) = \phi(x + 2\pi)$. Prove that ψ is also a solution.
- Show that ϕ is 2π -periodic if and only if $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$.
- Let ϕ_1, ϕ_2 be the solutions satisfying

$$\phi_1(0) = 1, \quad \phi_1'(0) = 0, \quad \phi_2(0) = 0, \quad \phi_2'(0) = 1.$$

Show that there exist constants a, b such that

$$\phi_1(x + 2\pi) = a \phi_1(x) + b \phi_2(x).$$

Solution.

- The coefficients $\cos x$ and $\sin x$ are 2π -periodic, so

$$\begin{aligned} \psi''(x) + \cos(x) \psi'(x) + \sin(x) \psi(x) &= \phi''(x + 2\pi) + \cos(x) \phi'(x + 2\pi) + \sin(x) \phi(x + 2\pi) \\ &= \phi''(x + 2\pi) + \cos(x + 2\pi) \phi'(x + 2\pi) + \sin(x + 2\pi) \phi(x + 2\pi) \\ &= 0. \end{aligned}$$

Thus ψ is a solution.

- (b) If ϕ is 2π -periodic then the two identities are immediate. Conversely, suppose $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$. Let $\psi(x) = \phi(x + 2\pi)$. By (a), ψ is a solution, and

$$\psi(0) = \phi(2\pi) = \phi(0), \quad \psi'(0) = \phi'(2\pi) = \phi'(0).$$

By uniqueness of the initial-value problem, $\psi \equiv \phi$, so ϕ is 2π -periodic.

- (c) The initial vectors $(1, 0)$ and $(0, 1)$ are linearly independent, so $\{\phi_1, \phi_2\}$ is a fundamental set. In particular the solution $\psi(x) = \phi_1(x + 2\pi)$ may be written uniquely as $\psi = a\phi_1 + b\phi_2$ for some constants a, b .

7. (a) Write the system

$$\begin{aligned} y_1' &= 4y_1 + 3y_2, \\ y_2' &= -y_1 \end{aligned}$$

in matrix form $y' = Ay$ with $A \in \mathbb{R}^{2 \times 2}$.

- (b) Find $P \in \mathbb{R}^{2 \times 2}$ such that $P^{-1}AP$ is diagonal.
(c) Compute e^{tA} .

Solution.

- (a)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

- (b) The matrix $A = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}$ has characteristic polynomial $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$, hence eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. The matrix of eigenvectors

$$P = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$$

therefore satisfies

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

- (c) Consequently

$$e^{tA} = P \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{-e^t + 3e^{3t}}{2} & \frac{-3e^t + 3e^{3t}}{2} \\ \frac{e^t - e^{3t}}{2} & \frac{3e^t - e^{3t}}{2} \end{pmatrix}.$$

8. Using the Abel–Liouville identity, show that

$$\Phi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a fundamental matrix for $y' = Ay$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution. Write $\Phi = (\Phi^1 \ \Phi^2)$ for the columns of Φ . Direct differentiation gives

$$(\Phi^1)' = A\Phi^1, \quad (\Phi^2)' = A\Phi^2,$$

so Φ is a solution matrix. Abel–Liouville yields

$$\det \Phi(t) = \det \Phi(t_0) \exp\left(\int_{t_0}^t \operatorname{tr} A(s) \, ds\right) = \det \Phi(t_0) \exp\left(\int_{t_0}^t 0 \, ds\right) = \det \Phi(t_0).$$

Taking $t_0 = 0$ gives $\det \Phi(t) = \det I = 1 \neq 0$. Hence $\Phi(t)$ is invertible for all t , and Φ is a fundamental matrix.

9. Solve the initial-value problem

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}, \quad x_1(0) = 1, \quad x_2(0) = -1.$$

Solution. The unique solution of $x' = Ax + g(t)$, $x(t_0) = \eta$, is

$$\varphi(t) = \Phi(t)\Phi(t_0)^{-1}\eta + \Phi(t) \int_{t_0}^t \Phi(s)^{-1}g(s) \, ds,$$

where Φ is any fundamental matrix. Taking $\Phi(t) = e^{tA}$ from Question 7 (so that $\Phi(0) = I$) and $\eta = (1, -1)^\top$ gives

$$\varphi(t) = e^{tA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{tA} \int_0^t e^{-sA} \begin{pmatrix} \sin s \\ 0 \end{pmatrix} \, ds.$$

Evaluating the integral produces

$$\varphi(t) = \begin{pmatrix} \frac{3}{20}e^{3t} + \frac{3}{4}e^t - \frac{1}{5}\sin t + \frac{1}{10}\cos t \\ -\frac{1}{20}e^{3t} - \frac{3}{4}e^t - \frac{1}{10}\sin t - \frac{1}{5}\cos t \end{pmatrix}.$$

(As a check: $\varphi(0) = (1, -1)^\top$.)