

MATE 4009 — Takehome Exam 2

Solutions

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1. Suppose $X: \mathbb{R} \rightarrow \mathbb{R}^3$ is linear. Show that there exist constants a, b, c such that

$$X(t) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} t, \quad t \in \mathbb{R}.$$

Solution. Let $X(1) = (a, b, c)^\top$. Linearity gives

$$X(t) = X(t \cdot 1) = t X(1) = t \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

2. (a) Are $\alpha(t) = t$ and $\beta(t) = t^{-1}$ linearly independent on $(0, \infty)$?
(b) Let $\{v, w, u\}$ be linearly independent. Show that $\{v, w\}$ is linearly independent.

Solution.

- (a) The Wronskian is

$$W(\alpha, \beta)(t) = \det \begin{pmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{pmatrix} = -\frac{1}{t} - \frac{1}{t} = -\frac{2}{t}.$$

Since $W \not\equiv 0$ on $(0, \infty)$, the pair is linearly independent there.

- (b) Suppose $c_1 v + c_2 w = \mathbf{0}$. Then $c_1 v + c_2 w + 0 \cdot u = \mathbf{0}$. Independence of $\{v, w, u\}$ forces $c_1 = c_2 = 0$. Hence $\{v, w\}$ is linearly independent.

*[Course pages](#)

3. (a) Let S be the solution set of $x' + p(t)x = q(t)$, with p, q continuous on an interval I . Fix $\hat{x} \in S$, and let K be the solution set of the homogeneous equation $x' + p(t)x = 0$. Show that

$$S = \{\hat{x} + \phi : \phi \in K\}.$$

- (b) Find a constant c such that $x_p(t) = c$ is a particular solution of

$$x' = \frac{t}{1-t^2} x + \frac{t}{1-t^2}.$$

Find the general solution on $(-1, 1)$. Sketch the direction field and the solutions with $x(0) \in \{-2, -1, 0, 1, 2\}$.

Solution.

- (a) This is merely a special case of the following translation principle for any linear equation was proven in the lectures.

Let $L : V \rightarrow W$ be any linear map. and x_p be a particular of a linear equation

$$Lx = b.$$

Then the set S of all solutions is the set

$$\begin{aligned} S &= x_p + \ker L \\ &= \{x_p + x_n : x_n \in \ker L\}. \end{aligned}$$

Indeed, if $Lx = b = Lx_p$ then $L(x - x_p) = 0$, so $x \in x_p + \ker L$. Conversely, if $x = x_p + x_n$ with $Lx_n = 0$, then $Lx = Lx_p + Lx_n = b$.

Here $L(x) = x' + px$, $b = q$, $\hat{x}$ plays the role of x_p , and $K = \ker L$.

- (b) By inspection, $x_p(t) = -1$ is a particular solution. Let ϕ be a solution of the homogeneous equation.

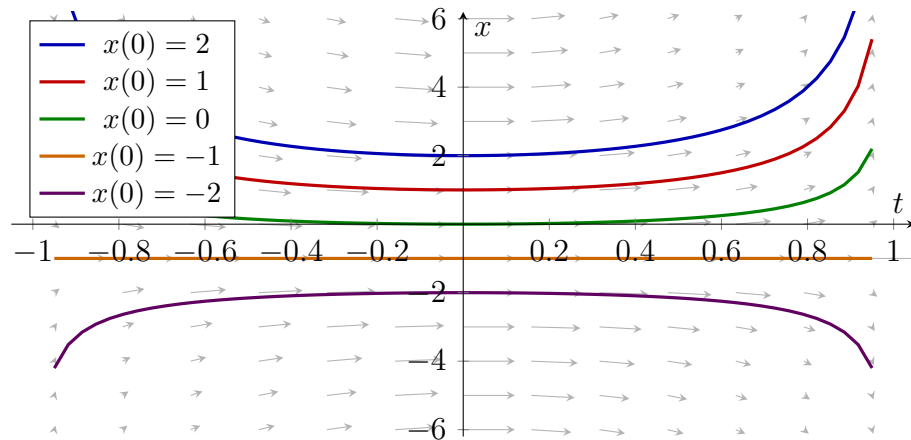
$$\begin{aligned} \int_0^t \frac{\phi'(\tau)}{\phi(\tau)} d\tau &= \int_0^t \frac{\tau}{1-\tau^2} d\tau \\ \int_{\phi_0}^{\phi(t)} \frac{1}{\xi} d\xi &= \int_0^t \frac{\tau}{1-\tau^2} d\tau \\ \ln \phi(t) - \ln \phi_0 &= -\frac{1}{2} \ln(1-t^2) \\ \phi(t) &= \frac{\phi_0}{\sqrt{1-t^2}}. \end{aligned}$$

Hence the general solutions of the have the form

$$x(t) = \phi(t) - 1 = \frac{\phi_0}{\sqrt{1-t^2}} - 1.$$

The condition $x(0) = x_0$ gives $\phi(0) = x_0 + 1$, hence

$$x(t) = \frac{x_0 + 1}{\sqrt{1-t^2}} - 1.$$



4. Let $p, q: [t_0, t_1] \rightarrow \mathbb{R}$ be continuous.

(a) Show that if ϕ solves $x' + p(t)x = q(t)$ and $P' = p$, then

$$\frac{d}{dt} [\phi(t)e^{P(t)}] = q(t)e^{P(t)}.$$

(b) Use (a) to solve $x' = x - t$.

Solution.

(a)

$$\frac{d}{dt} [\phi(t)e^{P(t)}] = \phi'(t)e^{P(t)} + p(t)\phi(t)e^{P(t)} = (\phi'(t) + p(t)\phi(t))e^{P(t)} = q(t)e^{P(t)}.$$

(b) Taking $p(t) = -1$ and $P(t) = -t$, the result of part (a) yields

$$\frac{d}{dt} x(t)e^{-t} = -te^{-t}.$$

Hence

$$\begin{aligned} x(t)e^{-t} &= - \int te^{-t} dt \\ &= e^{-t}(t+1) + C. \end{aligned}$$

and so

$$x(t) = (t+1) + Ce^t.$$

5. Find the value of x_0 for which the solution of

$$x' = x + \cos t, \quad x(0) = x_0$$

remains bounded as $t \rightarrow +\infty$.

Solution. Undetermined coefficients with $x_p = a \sin t + b \cos t$ produce

$$x_p(t) = \frac{1}{2}(\sin t - \cos t).$$

The homogeneous solutions are ce^t , so

$$x(t) = ce^t + \frac{1}{2}(\sin t - \cos t).$$

The initial condition gives $c = x_0 + \frac{1}{2}$, hence

$$x(t) = \left(x_0 + \frac{1}{2}\right)e^t + \frac{1}{2}(\sin t - \cos t).$$

The exponential term remains bounded on $[0, \infty)$ if and only if $x_0 = -\frac{1}{2}$, in which case

$$x(t) = \frac{1}{2}(\sin t - \cos t).$$

6.

7. Let $x(t)$ be the position of a ship of mass m at time t and x^* the position of a stationary iceberg. Suppose that after switching off the engines at time t_0 , the motion of the ship is governed by the equation

$$mx'' + 2\gamma x' = 0$$

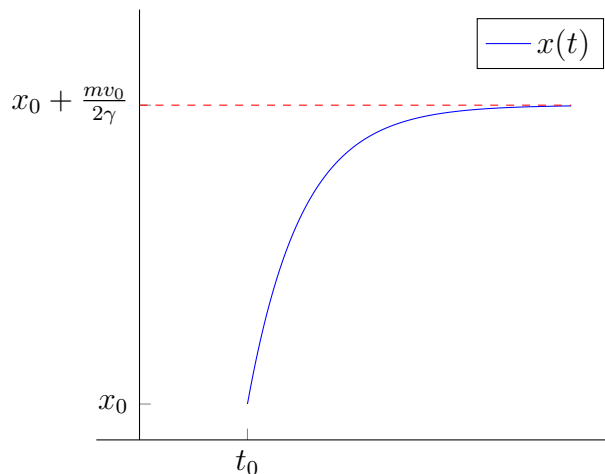
where $\gamma > 0$ is a constitutive constant. If $x(t_0) = x_0 < x^*$ and $x'(t_0) = v_0 > 0$, show that the ship will never reach the iceberg providing

$$\gamma > \frac{mv_0}{2(x^* - x_0)}.$$

Solution.

Let $v = x'$. Then

$$\begin{aligned} \frac{v'}{v} &= -2\frac{\gamma}{m}. \\ \implies \int_{t_0}^t \frac{v'(\tau)}{v(\tau)} d\tau &= -2\frac{\gamma}{m}(t - t_0) \\ \implies \int_{v_0}^{v(t)} \frac{1}{u} du &= -\frac{2\gamma}{m}(t - t_0) \\ \implies v(t) &= v_0 e^{-\frac{2\gamma}{m}(t-t_0)} \\ \implies \int_{t_0}^t v(\tau) d\tau &= \int_{t_0}^t v_0 e^{-\frac{2\gamma}{m}(\tau-t_0)} d\tau \\ \implies x(t) &= x_0 - \frac{m}{2\gamma} v_0 e^{-\frac{2\gamma}{m}(\tau-t_0)} \Big|_{t_0}^t \\ &= x_0 - \frac{m}{2\gamma} v_0 e^{-\frac{2\gamma}{m}(t-t_0)} + v_0 \frac{m}{2\gamma} \end{aligned}$$



Since, $v_0 > 0$ this is an increasing function bounded above by

$$\lim_{t \rightarrow \infty} x(t) = x_0 + v_0 \frac{m}{2\gamma}$$

and so $x(t) < x^*$ for all $t > t_0$ if and only if $\gamma > \frac{mv_0}{2(x^* - x_0)}$.

8. The displacement of a spring satisfies

$$x'' = -x \tag{1}$$

and the energy identity

$$(x')^2 + x^2 = 1. \tag{2}$$

A student concludes that $x' = \pm\sqrt{1-x^2}$ and studies only

$$x' = \sqrt{1-x^2}. \tag{3}$$

- (a) Find the general solution of (3).
- (b) Why do the solutions of (3) fail to look periodic, even though the spring motion is periodic?
- (c) What is the relation between solutions of (3) and the actual spring motion?
- (d) Draw a direction field and some solutions of

$$x' = -\sqrt{1-x^2}. \tag{4}$$

- (e) How did the non-physical solution $x(t) \equiv 1$ arise?

Solution.

- (a) On $(-1, 1)$ one has $dx/\sqrt{1-x^2} = dt$, so $x(t) = \sin(t - \phi)$ on every interval where the cosine (i.e. x') is nonnegative:

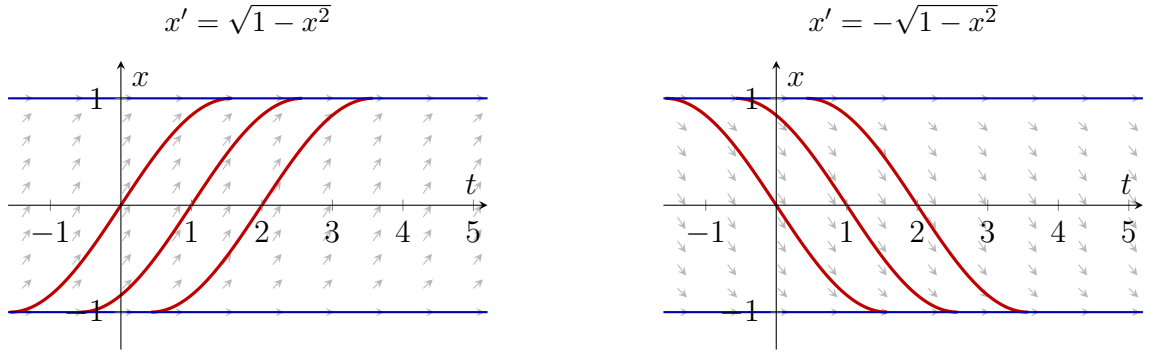
$$x(t) = \sin(t - \phi), \quad t \in \left(\phi - \frac{\pi}{2}, \phi + \frac{\pi}{2}\right).$$

Since $x' \geq 0$, solutions of (3) are nondecreasing. They may be continued by remaining at the equilibria ± 1 :

$$x(t) = \begin{cases} -1 & t \leq \phi - \pi/2, \\ \sin(t - \phi) & \phi - \pi/2 \leq t \leq \phi + \pi/2, \\ 1 & t \geq \phi + \pi/2. \end{cases}$$

In particular $x \equiv \pm 1$ are solutions, and every nonconstant maximal solution tends to 1 as $t \rightarrow +\infty$.

- (b) Equation (3) keeps only the positive square root, so it describes motion while $x' \geq 0$. A periodic oscillation must reverse at the turning points $x = \pm 1$ and follow the negative root (4) on the return trip. Restricting to one sign destroys periodicity and forces solutions to flatten at $x = 1$.
- (c) The orbits of (1) that lie on the energy ellipse (2) are concatenations of an increasing arc of (3) (from -1 to 1) with a decreasing arc of (4) (from 1 to -1). Each first-order equation captures half of one period.
- (d) Solutions of (4) are nonincreasing: $x(t) = -\sin(t - \phi)$ on intervals of length π , with optional sojourns at ± 1 . Every nonconstant maximal solution tends to -1 as $t \rightarrow +\infty$.



- (e) Differentiating the energy identity along a C^2 curve gives $x'(x'' + x) = 0$. Wherever $x' \neq 0$ one recovers (1). At a point where $x' = 0$ and $x = \pm 1$, the factor x' vanishes and the reduction $x' = \pm\sqrt{1-x^2}$ admits the constants $x \equiv \pm 1$, even though

$$\frac{d^2}{dt^2}(1) + 1 = 1 \neq 0.$$

Thus $x \equiv 1$ solves the first-order equation (3) and lies on the energy level set (2), but it is *not* a solution of the original oscillator (1). It appears because solving (2) for x' is algebraically valid at the turning points, while the second-order dynamics are not.

9. (a) A function $F: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is *zoom invariant* if there exists $\lambda > 0$ such that $F(\lambda t, \lambda x) = F(t, x)$ for all $(t, x) \in D$. Show that

$$F(t, x) = \frac{x^2 + 2xt}{t^2}$$

is zoom invariant (on $t \neq 0$).

- (b) If F is zoom invariant and m solves

$$m' = \frac{F(1, m) - m}{t}, \quad m(t_0) = \frac{x_0}{t_0}, \quad (5)$$

show that $x(t) = tm(t)$ solves $x' = F(t, x)$, $x(t_0) = x_0$.

- (c) Solve $x' = (x^2 + 2xt)/t^2$.

Solution.

- (a)

$$F(\lambda t, \lambda x) = \frac{(\lambda x)^2 + 2(\lambda x)(\lambda t)}{(\lambda t)^2} = \frac{x^2 + 2xt}{t^2} = F(t, x).$$

- (b) Zoom invariance with scale factor t gives $F(t, tm) = F(1, m)$. Hence if $x(t) = tm(t)$,

$$x' = m + tm' = m + F(1, m) - m = F(1, m) = F(t, tm) = F(t, x),$$

and $x(t_0) = t_0 m(t_0) = x_0$.

- (c) Here $F(1, m) = m^2 + 2m$, so

$$m' = \frac{m(m+1)}{t}.$$

If $m \neq 0$ and $m \neq -1$,

$$\int_{\mu_0}^{m(t)} \left(\frac{1}{\mu} - \frac{1}{\mu+1} \right) d\mu = \int_{t_0}^t \frac{d\tau}{\tau},$$

hence

$$\frac{m(t)}{m(t)+1} = \frac{\mu_0}{\mu_0+1} \cdot \frac{t}{t_0}, \quad m(t) = \frac{\mu_0 t}{\mu_0(t_0 - t) + t_0}.$$

Therefore

$$x(t) = \frac{\mu_0 t^2}{\mu_0(t_0 - t) + t_0}.$$

If $\mu_0 \neq 0$ this is of the form $x(t) = t^2/(c - t)$. The excluded value $\mu_0 = -1$ is the additional solution $m \equiv -1$, i.e. $x(t) = -t$. The value $\mu_0 = 0$ is the solution $x \equiv 0$.

10. An undamped spring with periodic forcing satisfies

$$x'' + \omega_0^2 x = F \cos(\omega t), \quad (6)$$

with $\omega \neq \omega_0$.

- Find the general solution of the homogeneous equation.
- Find a particular solution of (6) by undetermined coefficients.
- Write the general solution of (6).
- Solve the initial-value problem $x(0) = x'(0) = 0$.
- Describe the motion when $|\omega_0 - \omega|$ is small compared with $\omega_0 + \omega$.

Solution. The homogeneous equation $x'' + \omega_0^2 x = 0$ has general solution

$$c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

Trying $\hat{x}(t) = A \cos(\omega t)$ in (6) gives $A = F/(\omega_0^2 - \omega^2)$, so

$$\hat{x}(t) = \frac{F \cos(\omega t)}{\omega_0^2 - \omega^2}.$$

The general solution is therefore

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F \cos(\omega t)}{\omega_0^2 - \omega^2}.$$

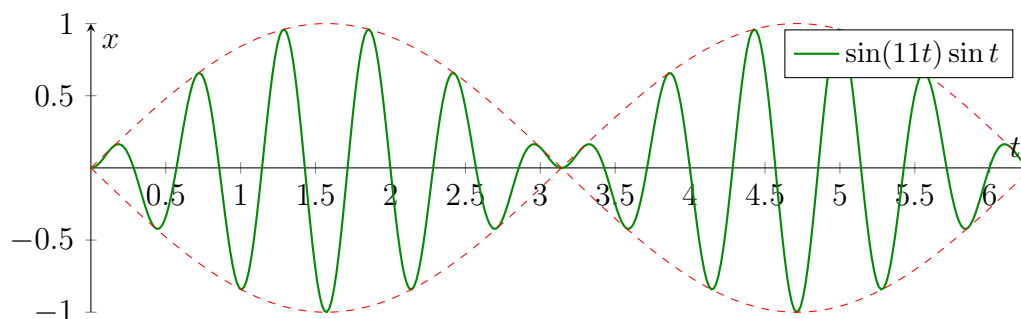
The conditions $x(0) = x'(0) = 0$ yield $c_2 = 0$ and $c_1 = -F/(\omega_0^2 - \omega^2)$. Writing $A = F/(\omega_0^2 - \omega^2)$,

$$x(t) = A(\cos \omega t - \cos \omega_0 t).$$

The sum-to-product identity $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$ rearranges this as

$$x(t) = 2A \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

If $|\omega_0 - \omega| \ll \omega_0 + \omega$, this is a rapid oscillation of angular frequency $(\omega_0 + \omega)/2$ whose amplitude is modulated at the slow beat frequency $|\omega_0 - \omega|/2$.



Beats for $\omega_0 = 12$, $\omega = 10$ (carrier frequency 11, envelope frequency 1).

- Define lower and upper fences.
 - Sketch the isoclines of $x' = x^2 - t$ for slopes 0 and -1 .
 - Show that $l(t) = -\sqrt{t}$, $t > 0$, is a lower fence. What does this imply for solutions?
 - Show that $u(t) = -\sqrt{t-1}$, $t > 5/4$, is an upper fence.
 - Show that $u(t) - l(t) \rightarrow 0$ as $t \rightarrow \infty$.

Solution.

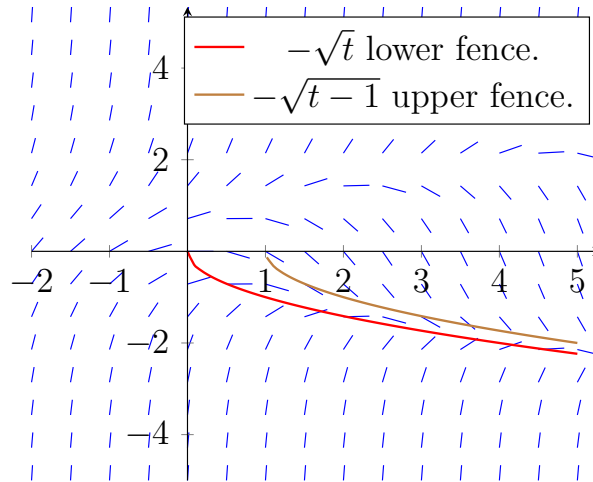
- (a) Let $F: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and $a < b$. A C^1 function $l: (a, b) \rightarrow \mathbb{R}$ is a *lower fence* for F if $l'(t) < F(t, l(t))$ on (a, b) . A C^1 function $u: (a, b) \rightarrow \mathbb{R}$ is an *upper fence* if $F(t, u(t)) < u'(t)$ on (a, b) .
- (b) The isocline of slope c is $x^2 - t = c$, i.e. $x^2 = t + c$. Thus slope 0 is the parabola $x^2 = t$, and slope -1 is $x^2 = t - 1$.
- (c) Let $F(t, x) = x^2 - t$. Then $l'(t) = -1/(2\sqrt{t})$ and $F(t, l(t)) = 0$, so $l' < F(t, l)$ holds for all $t > 0$. Solutions cannot cross l downwards: if $x(t_1) > l(t_1)$, then $x(t) > l(t)$ for all later t in the interval of existence.
- (d) Now $F(t, u(t)) = -1$ and $u'(t) = -1/(2\sqrt{t-1})$, so

$$-1 < -\frac{1}{2\sqrt{t-1}} \iff \sqrt{t-1} > \frac{1}{2} \iff t > \frac{5}{4}.$$

(e)

$$u(t) - l(t) = \sqrt{t} - \sqrt{t-1} = \frac{1}{\sqrt{t} + \sqrt{t-1}} \rightarrow 0 \quad (t \rightarrow \infty).$$

Thus on $t > 5/4$ the pair (l, u) is a funnel: any solution that enters the strip $l(t) < x(t) < u(t)$ remains there and is squeezed toward $-\sqrt{t}$.



A funnel in the direction field of $x' = x^2 - t$