MATE 4009 – Exam 1 – Practice Problems

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1. Find a differential equation whose solution set is a one parameter family consisting of functions of the form

$$\phi_c(x) = c \tan x.$$

Solution.

It is required to find $f(x, y)$ such that the ODE $y' = f(x, y)$ has solutions of the form $\phi_c(x) = c \tan x$.

$$\begin{aligned}
& y = c \tan x \\
\implies & \frac{y}{\tan x} = c \\
\implies & \frac{y' \tan x - y \tan' x}{\tan^2 x} = 0 \\
\implies & y' \tan x - y \sec^2 x = 0 \\
\implies & y' = \frac{y}{\sin x \cos x} \\
\implies & y' = \frac{2y}{\sin 2x}.
\end{aligned}$$

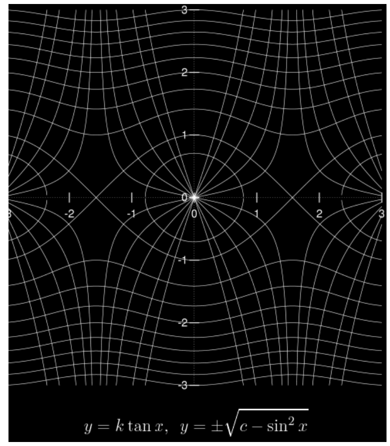
2. Find a family of functions, orthogonal to the family $\phi_c(x) = c \tan x$. Sketch some members of each family.

Solution.

Orthogonal solutions must satisfy the separable ODE $y' = -\frac{\sin 2x}{2y}$. Let $y(x)$ be a solution.

$$\begin{aligned}
& 2yy' = -\sin 2x \\
\implies & \frac{d}{dx} y'^2 = -\sin 2x \\
\implies & y'^2 = \frac{\cos 2x}{2} + \text{ a constant} \\
\implies & y'^2 = C - \sin^2 x.
\end{aligned}$$

Where C is constant. Hence $\phi_C(x) = \pm\sqrt{C - \sin^2 x}$ is an orthogonal family.



3. For a chemical reaction of the form $\mathrm{A} \longrightarrow \mathrm{mM} + \mathrm{nN}$, let θ be the concentration of reactant A at time t. Let $[A]_0$ denote the initial concentration at time t_0. If $[A]_t$ satisfies the second order reaction equation $\frac{d[\mathrm{A}]_t}{dt} = -k[A]^2$, show that

$$[\mathrm{A}]_t = \frac{[\mathrm{A}]_0}{1 + [\mathrm{A}]_0 k(t - t_0)}, \qquad t \geq t_0.$$

Show that any other solution starting at t_0 is merely a restriction.

Solution.

$$\begin{aligned}
& \int_{t_0}^{t} \frac{[\mathrm{A}]'_t}{[\mathrm{A}]_t^2}\, d\tau = -k \int_{t_0}^{t} 1\, d\tau \\
\implies & \int_{[\mathrm{A}]_0}^{[\mathrm{A}]_t} \frac{1}{\xi^2}\, d\xi = -k(t - t_0) \\
\implies & -\frac{1}{[\mathrm{A}]_t} + \frac{1}{[\mathrm{A}]_0} = -k(t - t_0) \\
\implies & \frac{1}{[\mathrm{A}]_t} = \frac{1}{[\mathrm{A}]_0} + k(t - t_0) \\
\implies & [\mathrm{A}]_t = \frac{[\mathrm{A}]_0}{1 + [\mathrm{A}]_0 k(t - t_0)}.
\end{aligned}$$

See class notes on autonomous ODE's for proof of uniqueness at a non singular point.

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4. Find an infinite number of solutions of the initial value problem (IVP)

$$x' = 2\sqrt{x}, \quad x(0) = 0.$$

Sketch the slope field of the differential equation and in the same drawing graph some solutions of the IVP.

Solution.

Since $x = 0$ is a singular point $\phi(t) = 0, \quad t \in \mathbb{R}$ is a solution. If $x_0 \neq 0$ separation of variables yields

$$\phi(t) = (t - t_0 + \sqrt{x_0})^2,$$

i.e., a translate of the parabola $x = t^2$. For each $C > 0$ define

$$\phi_C(t) = \begin{cases} (t-C)^2 & \text{if } t \geq C \ , \\ 0 & \text{if } t \leq C \end{cases}$$

The functions ϕ_C with $C \geq 0$ comprise a one parameter family of solutions of the IVP.

5. A point of mass m moving along a line experiences a force F which is a function of its position. Let $x(t)$ be the position of the point at time t. According to Newton's law (first written as a differential equation by Euler), the equation of motion is:

$$mx'' = F(x). \tag{1}$$

   (a) Suppose that the force F has an antiderivative $-\phi$ (so that, $\phi' = -F$). Show that if $x(t)$ is a solution of (1), then the quantity

   $$E = \frac{m}{2}x''(t)^2 + \phi(x(t))$$

   is a constant.

   (b) Show that (1), subject to initial condition $x(t_0) = x_0$, has local solutions $x(t)$ given implicitly by the equation:

   $$\pm\sqrt{\frac{m}{2}} \int_{x_0}^{x(t)} \frac{1}{\sqrt{E - \phi(x)}} dx = \int_{t_0}^{t} dt \tag{2}$$

Solution.

Let $x : I \to \mathbb{R}$ be a solution on an interval I.

$$\begin{aligned} & mx''(t) = f(x(t)), \quad t \in I \\ \implies & mx''(t) = -\phi'(x(t)) \quad t \in I \\ \implies & mx'x'' - \phi'(x(t))x'(t) = 0 \quad t \in I \\ \implies & \frac{d}{dt}\left(\frac{1}{2}mx'^2 + \phi(x(t))\right) = 0 \quad t \in I. \end{aligned}$$

By the Mean Value Theorem, it must be that

$$E = \frac{m}{2}x''(t)^2 + \phi(x(t))$$

is a constant on I. Solving this equation for x' gives two autonomous ODE's

$$x' = \pm\sqrt{\tfrac{2}{m}}\sqrt{E - \phi(x)}.$$

Equation (2) follows immediately by separation of variables.

6. A function $F : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called *separable on D* if there exist functions f, g such that $F(x, y) = f(x)g(y)$ for all $(x, y) \in D$. Let F be separable with partial derivatives D_1F, D_2F D_1D_2F exist on D. Show that $FD_1D_2F = (D_1F)(D_2F)$.

Solution.

Suppose that $F(x, y) = f(x)g(y)$. Then,

$$\begin{aligned} D_1F(x,y) &= f'(x)g(y) \\ D_2F(x,y) &= f(x)g'(y) \\ D_1D_2(x,y) &= f'(x)g'(y) \\ FD_1D_2F(x,y)) &= f(x)g(y) \\ (D_1F)(D_2F)(x,y) &= f'(x)g(y)f(x)g'(y). \end{aligned}$$

Hence, $FD_1D_2F = (D_1F)(D_2F)$.

7. Solve the initial value problem

$$y' = -x/y, \quad y(x_0) = \sqrt{3}.$$

Sketch some maximal solutions of the differential equation in the case $x_0 < 0$.

Solution.

If ϕ is a solution,

$$\begin{aligned}
\phi'(x) &= \frac{-x}{\phi(x)} \\
\implies \int_{x_0}^{x} \phi(\xi)\phi'(\xi)d\xi &= -\int_{x_0}^{x} \xi d\xi \\
\implies \int_{\phi(x_0)}^{\phi(x)} u\, du &= -\int_{x_0}^{x} \xi d\xi \\
\implies \phi(x)^2 - \phi(x_0)^2 &= x_0^2 - x^2 \\
\implies \phi_\pm(x) &= \pm\sqrt{\phi(x_0)^2 + x_0^2 - x^2}
\end{aligned}$$

If $x_0 < 0$, solutions are semicircles in the lower half plane.

$$y = -\sqrt{r^2 - x^2}, \quad -r < x < r.$$

The solutions do not extend to $x = r$ since the point $(r, 0)$ is not in the domain of the ODE.

8. The position, $x(t)$ of a falling body of mass m satisfies the IVP

$$\begin{cases} m\ddot{x} = gm - \gamma\dot{x}^2 \\ x(t_0) = x_0. \\ x'(t_0) = 0 \end{cases}$$

where g is the gravitational constant and γ is a constitutive constant representing atmospheric retardation (drag). Find scalings of space and time which reduces this equation to the dimensionless form

$$\chi'' = 1 - (\chi')^2.$$

Hence, or otherwise, show that

$$x(t) = x_0 + \frac{m}{\gamma}\log\left[\cosh(\sqrt{\frac{\gamma g}{m}}(t - t_0)\right].$$

Solution.

Let $t_c, x_c > 0$ and let $x(t)$ be a solution of (5). Define

$$\chi(\tau) = \frac{1}{x_c} \cdot x(t_c\tau).$$

so that

$$\begin{aligned}
\chi'(\tau) &= \frac{t_c}{x_c} \cdot x'(t_c\tau) \\
\chi''(\tau) &= \frac{t_c^2}{x_c} \cdot x''(t_c\tau) \\
&= \frac{t_c^2}{x_c}\left[g - \frac{\gamma}{m}\left[x'(t_c\tau)\right]^2\right] \\
&= \frac{t_c^2 g}{x_c} - \frac{\gamma x_c}{m}\left[\chi'(\tau)\right]^2
\end{aligned}$$

Choosing $x_c = \frac{m}{\gamma}$ and $t_c = \sqrt{\frac{m}{g\gamma}}$ yields the dimensionless equation:

$$\chi'' = 1 - (\chi')^2$$

and the initial conditions become

$$\begin{cases} \chi(t_0) = \dfrac{x_0}{x_c} \\ \chi'(t_0) = 0. \end{cases}$$

Solving for χ' by separation of variables yields

$$\chi'(\tau) = \tanh(\tau - \tau_0).$$

Integrating over the interval $[\tau_0, \tau]$ gives

$$\chi(\tau) = \chi(\tau_0) + \log(\cosh(\tau - \tau_0)).$$

Rewriting in terms of the variables x and t:

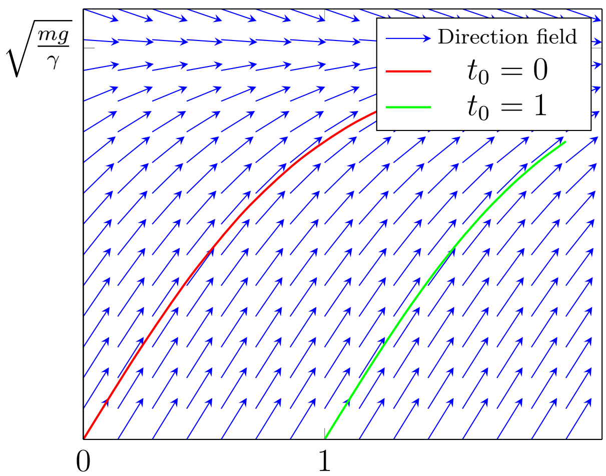
$$\frac{x(t)}{x_c} = \frac{x(t_0)}{x_c} + \log\left[\cosh(\frac{t}{t_c} - \frac{t_0}{t_c})\right].$$

Hence,

$$x(t) = x_0 + \frac{m}{\gamma}\log\left[\cosh(\sqrt{\frac{\gamma g}{m}}(t - t_0)\right].$$

Remark. This is easily inverted to yield the time of flight as a function of position.

The diagram below shows $v(t)$ for $t_0 = 0$, and $t_0 = 1$. Notice that solutions approach a terminal velocity of $\sqrt{\frac{mg}{\gamma}}$.



9. Let a be a constant and q a real valued function continuous on an interval I. Verify that

$$\phi(t) = x_0 e^{at} + e^{at} \int_{t_0}^{t} e^{-as} q(s)\, ds, \quad t \in R. \tag{3}$$

is a solution of the IVP

$$\begin{cases} x' = ax + q(t), & (t,x) \in \mathbb{R}^2 \\ x(t_0) = x_0. \end{cases} \tag{4}$$

Show that any other solution

$$\psi : I \to \mathbb{R},$$

of the ODE (4), where I is an interval, is a restriction of that given by (3).

Solution.

Substituting (exercise) in the ODE shows that the function ϕ is indeed a solution.

Let $\psi : I \to \mathbb{R}$ be any solution of the IVP. Then

$$(\psi - \phi)' = a(\psi - \phi)$$
$$(\psi - \phi)(t_0) = 0$$

Thus $x = \psi - \phi$ is a solution of the ODE

$$\begin{cases} x' = ax \\ x(t_0) = 0. \end{cases} \tag{5}$$

To show that $\psi = \phi$ we need only show that this has unique solution $x(t) = 0, \quad t \in \mathbb{R}$. Clearly, $x(t) = 0, \quad t \in \mathbb{R}$ is a solution. If $\xi : I \to \mathbb{R}$ is any solution of (5), then

$$\begin{aligned} \frac{d}{dt}\xi e^{-at} &= (\xi' - a\xi)e^{-at} \\ &= 0. \end{aligned}$$

Hence, by the mean value theorem, ξe^{-at} is constant on I. But, the initial condition $\xi(t_0) = 0$ means that $\xi e^{-at} = 0, \quad t \in I$. Since $e^{-at} > 0$, it must be that $\xi(t) = 0, \quad t \in I$ which means that ξ is merely a restriction of the zero solution.

10. In the interior of a homogeneous spherical planet, the gravitational force on a mass m, located at distance x from the centre, is $F(x) = -kmx$ where $k > 0$ is a constant. A tunnel is constructed from pole to pole and a ball of mass m is placed in the tunnel. Assume that there is no air resistance or friction or other force (occult or otherwise) within the tunnel.

    (a) Assuming Newton's law, write an initial value problem (IVP) to determine the motion when the ball is released from rest at the center of the earth ($x = 0$) at time $t = 0$. Show that the unique solution of this IVP is the zero function on $\mathbb{R}$.

    (b) Find the position of the ball as a function of time when dropped from a pole at time $t = 0$. Find also, the velocity of the ball when $x = 0$ and the time it takes to go from pole to center.

    (c) A straight tunnel is drilled through the earth from London to Moscow and another from London to New York. A tube is dropped in the tunnel to Moscow at time

$t = 0$ and travels under the influence only of gravity towards Moscow. Another tube is dropped simultaneously in the tunnel to New York. Assuming as before that there is no air resistance or friction or other force (apart from gravity) within the tunnels show that the tubes arrive at their respective destinations and moreover at the same time.

Solution.

By Newton's law

$$mx'' = -kmx$$

We must solve the IVP

$$\begin{aligned} x'' + kx &= 0, \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

$$\begin{aligned} & x'' + kx = 0, \\ \implies & x'x'' + kx'x = 0 \\ \implies & \frac{d}{dt}\left(\frac{x'^2}{2} + k\frac{x^2}{2}\right) = 0 \end{aligned}$$

By the mean value theorem, it must be that the quantity

$$x'^2 + kx^2$$

is constant. Putting $t = 0$, and using the the initial conditions, it is seen that

$$x'^2 + kx^2 = 0$$

Since $k > 0$ this can only happen if $x(t) = x'(t) = 0, \quad t \in \mathbb{R}$. I.e., the ball remains in equilibrium at the center of the planet.

In part (b) we must solve

$$\begin{aligned} x'' + kx &= 0, \\ x(0) &= R \\ x'(0) &= 0 \end{aligned}$$

where R is the radius of the planet. Using the procedure given in Example 5 of the class notes on nondimensionalization, the solution is

$$\begin{aligned} x(t) &= x(0)\cos\sqrt{k}\ t + x'(0)\sin\sqrt{k}\ t \\ &= R\cos\sqrt{k}\ t \end{aligned}$$

The velocity is

$$x'(t) = -\sqrt{k}R\sin\sqrt{k}\ t$$

Notice that $x = 0$ when $t = \dfrac{\pi}{2\sqrt{k}}$. This is the time required to go from the pole to the center. At this time the velocity is

$$x'(\pi/2\sqrt{k}) = -\sqrt{k}R$$

Remark. This also follows from conservation of energy.

In part (c) we must apply Newton's law to the projection of the force along the tunnels. Consider any tunnel of length L. Let ξ be the position of a point mass in the tunnel at distance ξ from its center and let x be the distance of the mass to the center of the sphere. Then $\xi = x\cos\theta$ where θ is the angle between the tunnel and the line segment from the center of the sphere to the mass. Applying Newton's law to the projection of the force along the tunnel, we have,

$$\begin{aligned} & F = -kmx \\ \implies & F\cos\theta = -kmx\cos\theta \\ \implies & F\cos\theta = -km\xi \\ \implies & m\xi'' = -km\xi \\ \implies & \xi'' = -k\xi \end{aligned}$$

The solution is

$$\xi(t) = \xi(0)\cos\sqrt{k}\ t + \xi'(0)\sin\sqrt{k}\ t.$$

The initial conditions are now $\xi(0) = \dfrac{L}{2}$ and $\xi'(0) = 0$, and so

$$\xi(t) = \frac{L}{2} \cos \sqrt{k}\ t.$$

It follows that the period of oscillation $\frac{2\pi}{\sqrt{k}}$ is independent of the length of the tunnel. and since the amplitude is $L/2$, a point mass starting a $\xi = L/2$ with initial velocity 0, will arrive at $\xi = -L/2$ in one half of the period.