

MATE 4009 — Takehome Exam 1

Solutions

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1. Find a differential equation whose solution set is the one-parameter family $\phi_c(x) = c \tan x$.

Solution. We seek f such that $y' = f(x, y)$ has solutions $y = c \tan x$. Eliminating the parameter c ,

$$\begin{aligned}y &= c \tan x \\ \implies \frac{y}{\tan x} &= c \\ \implies \frac{y' \tan x - y \sec^2 x}{\tan^2 x} &= 0 \\ \implies y' \tan x - y \sec^2 x &= 0 \\ \implies y' &= \frac{y}{\sin x \cos x} = \frac{2y}{\sin 2x}.\end{aligned}$$

2. Find a family of curves orthogonal to $\phi_c(x) = c \tan x$. Sketch some members of each family.

Solution. From Problem 1 the original family satisfies $y' = 2y/\sin 2x$. Orthogonal trajectories therefore satisfy

$$y' = -\frac{\sin 2x}{2y}.$$

If $y \neq 0$,

$$\begin{aligned}2yy' &= -\sin 2x \\ \implies \frac{d}{dx}(y^2) &= -\sin 2x \\ \implies y^2 &= \frac{\cos 2x}{2} + C = K - \sin^2 x,\end{aligned}$$

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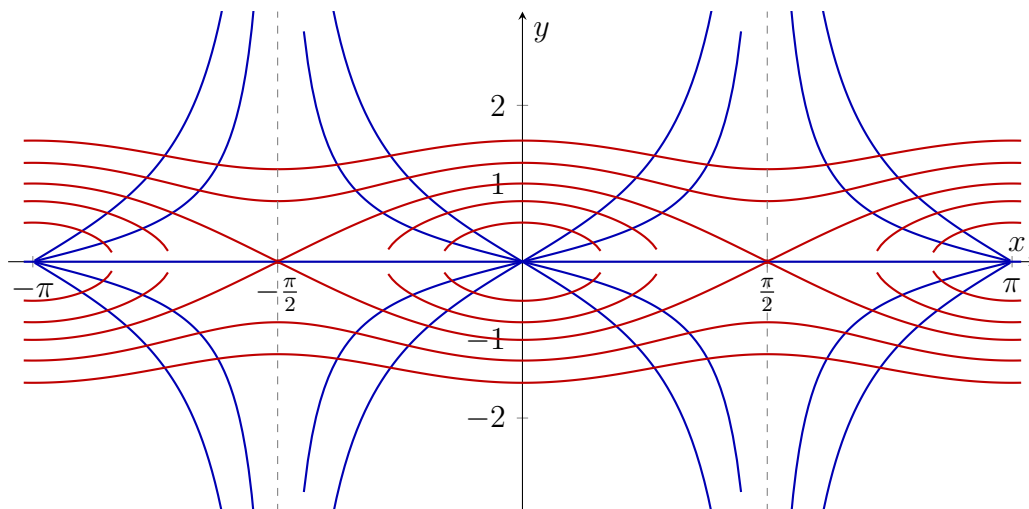
where $K = C + \frac{1}{2}$. Hence the orthogonal family is

$$\phi_K(x) = \pm \sqrt{K - \sin^2 x},$$

defined for those x with $\sin^2 x \leq K$.

- $0 < K < 1$: real only on closed subintervals where $|\sin x| \leq \sqrt{K}$ (a chain of oval arcs).
- $K = 1$: $y = \pm \cos x$ (up to consistent choice of sign on each interval).
- $K > 1$: defined for all x , a pair of 2π -periodic waves.

The figure shows members of $y = c \tan x$ (blue) and of $y^2 + \sin^2 x = K$ (red). Crossing curves meet at right angles.



$$\text{---} y = c \tan x \quad \text{---} y^2 + \sin^2 x = K$$

3. For a reaction $A \longrightarrow mM + nN$, let $[A]_t$ be the concentration of reactant A at time t , and let $[A]_0$ be the concentration at time t_0 . If $[A]_t$ satisfies the second-order rate equation

$$\frac{d}{dt}[A]_t = -k[A]_t^2,$$

show that

$$[A]_t = \frac{[A]_0}{1 + [A]_0 k(t - t_0)}, \quad t \geq t_0.$$

Show that any other solution starting at t_0 is a restriction of this one.

Solution. If $[A]_t \neq 0$,

$$\begin{aligned} \int_{t_0}^t \frac{[A]_\tau'}{[A]_\tau^2} d\tau &= -k \int_{t_0}^t 1 d\tau \\ \implies \int_{[A]_0}^{[A]_t} \frac{1}{\xi^2} d\xi &= -k(t - t_0) \\ \implies -\frac{1}{[A]_t} + \frac{1}{[A]_0} &= -k(t - t_0) \\ \implies [A]_t &= \frac{[A]_0}{1 + [A]_0 k(t - t_0)}. \end{aligned}$$

The initial-value problem has a unique solution through each $(t_0, [A]_0)$. The formula above is that unique solution (and reduces to the zero solution if $[A]_0 = 0$). Any other solution defined on a subinterval of the maximal interval is therefore a restriction of it.

4. Find infinitely many solutions of the initial-value problem

$$x' = 2\sqrt{x}, \quad x(0) = 0.$$

Sketch the slope field and graph some of these solutions.

Solution. $x = 0$ is an equilibrium: $\phi(t) = 0$ for all $t \in \mathbb{R}$ is a solution.

For $x > 0$, separation of variables gives

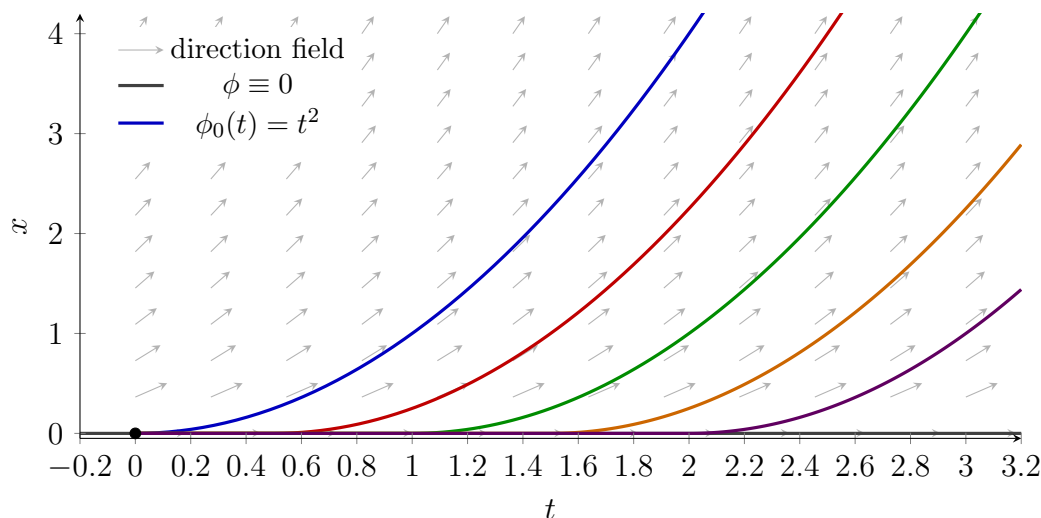
$$\phi(t) = (t - t_0 + \sqrt{x_0})^2.$$

Piecing the equilibrium together with a rightward parabola produces, for each $C \geq 0$,

$$\phi_C(t) = \begin{cases} 0 & t \leq C, \\ (t - C)^2 & t \geq C. \end{cases}$$

Each ϕ_C is C^1 , satisfies $x' = 2\sqrt{x}$ everywhere, and obeys $\phi_C(0) = 0$. This is an infinite (one-parameter) family of solutions of the IVP.

The figure shows the direction field of $x' = 2\sqrt{x}$ together with $\phi \equiv 0$ and the envelope solutions ϕ_C for $C = 0, 0.5, 1, 1.5, 2$. Every pictured integral curve passes through the origin at time $t = 0$.



5. A point mass m moving along a line is subject to a force F depending only on position. If $x(t)$ is the position at time t , Newton's law reads

$$mx'' = F(x). \quad (1)$$

- (a) Suppose F has an antiderivative $-\phi$ (i.e. $\phi' = -F$). Show that if x solves (1), then

$$E = \frac{m}{2}(x'(t))^2 + \phi(x(t))$$

is constant.

- (b) Show that (1) with $x(t_0) = x_0$ has local solutions given implicitly by

$$\pm \sqrt{\frac{m}{2}} \int_{x_0}^{x(t)} \frac{d\xi}{\sqrt{E - \phi(\xi)}} = \int_{t_0}^t d\tau. \quad (2)$$

Solution. Let $x: I \rightarrow \mathbb{R}$ be a solution on an interval I . Then

$$\begin{aligned} mx''(t) &= F(x(t)) = -\phi'(x(t)) \\ \implies mx'(t)x''(t) + \phi'(x(t))x'(t) &= 0 \\ \implies \frac{d}{dt} \left(\frac{m}{2}(x'(t))^2 + \phi(x(t)) \right) &= 0. \end{aligned}$$

A function whose derivative vanishes on an interval is constant, so E is constant on I .

Solving $E = \frac{m}{2}(x')^2 + \phi(x)$ for the velocity yields the two autonomous equations

$$x' = \pm \sqrt{\frac{2}{m}} \sqrt{E - \phi(x)}.$$

Separation of variables gives (2).

6. A function $F: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is *separable on D* if $F(x, y) = f(x)g(y)$ for all $(x, y) \in D$. Assume F is separable and that the partial derivatives D_1F , D_2F , and D_1D_2F exist on D . Show that

$$F D_1D_2F = (D_1F)(D_2F).$$

Solution. Write $F(x, y) = f(x)g(y)$. Then

$$\begin{aligned}D_1F(x, y) &= f'(x)g(y), \\D_2F(x, y) &= f(x)g'(y), \\D_1D_2F(x, y) &= f'(x)g'(y),\end{aligned}$$

and therefore

$$\begin{aligned}F D_1D_2F &= f(x)g(y) f'(x)g'(y), \\(D_1F)(D_2F) &= f'(x)g(y) f(x)g'(y).\end{aligned}$$

The two sides agree on D .

7. Solve

$$y' = -\frac{x}{y}, \quad y(x_0) = \sqrt{3}.$$

Sketch some maximal solutions in the case $x_0 < 0$.

Solution. If ϕ is a solution with $\phi \neq 0$,

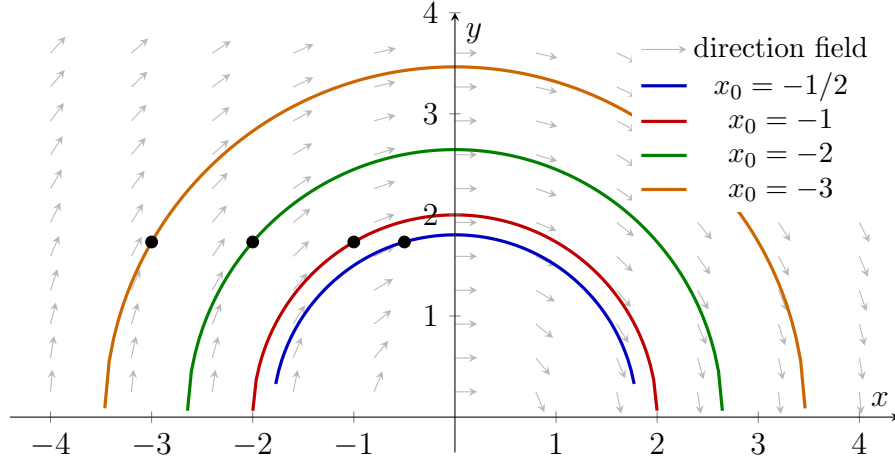
$$\begin{aligned}\phi'(x) &= -\frac{x}{\phi(x)} \\ \implies \int_{x_0}^x \phi(\xi)\phi'(\xi) \, d\xi &= -\int_{x_0}^x \xi \, d\xi \\ \implies \int_{\phi(x_0)}^{\phi(x)} u \, du &= -\int_{x_0}^x \xi \, d\xi \\ \implies \phi(x)^2 - \phi(x_0)^2 &= x_0^2 - x^2 \\ \implies \phi(x) &= \pm \sqrt{\phi(x_0)^2 + x_0^2 - x^2}.\end{aligned}$$

The initial condition $\phi(x_0) = \sqrt{3} > 0$ selects the positive branch:

$$\phi(x) = \sqrt{3 + x_0^2 - x^2}.$$

This is the upper semicircle of radius $r = \sqrt{3 + x_0^2}$ centred at the origin. The maximal interval is $(-r, r)$: the endpoints $(\pm r, 0)$ do not lie in the domain of $f(x, y) = -x/y$.

The figure shows the direction field and four maximal solutions for $x_0 = -0.5, -1, -2, -3$, each an upper semicircle through the point $(x_0, \sqrt{3})$. Solutions stop at the x -axis.



8. The position $x(t)$ of a falling body of mass m satisfies

$$\begin{cases} m\ddot{x} = mg - \gamma\dot{x}^2, \\ x(t_0) = x_0, \\ \dot{x}(t_0) = 0, \end{cases} \quad (3)$$

where g is the gravitational constant and $\gamma > 0$ is a drag coefficient. Find scalings of space and time that reduce the equation to the dimensionless form

$$\chi'' = 1 - (\chi')^2.$$

Hence, or otherwise, show that

$$x(t) = x_0 + \frac{m}{\gamma} \log \left[\cosh \left(\sqrt{\frac{\gamma g}{m}} (t - t_0) \right) \right].$$

Solution. Let $t_c, x_c > 0$ and set $\chi(\tau) = x(t_c\tau)/x_c$. Then

$$\begin{aligned} \chi'(\tau) &= \frac{t_c}{x_c} x'(t_c\tau), \\ \chi''(\tau) &= \frac{t_c^2}{x_c} x''(t_c\tau) = \frac{t_c^2}{x_c} \left(g - \frac{\gamma}{m} [x'(t_c\tau)]^2 \right) \\ &= \frac{t_c^2 g}{x_c} - \frac{\gamma x_c}{m} [\chi'(\tau)]^2. \end{aligned}$$

The choices

$$x_c = \frac{m}{\gamma}, \quad t_c = \sqrt{\frac{m}{g\gamma}}$$

yield $\chi'' = 1 - (\chi')^2$, with initial conditions $\chi(\tau_0) = x_0/x_c$ and $\chi'(\tau_0) = 0$, where $\tau_0 = t_0/t_c$.

Separation of variables for χ' gives $\chi'(\tau) = \tanh(\tau - \tau_0)$. Integrating over $[\tau_0, \tau]$ produces

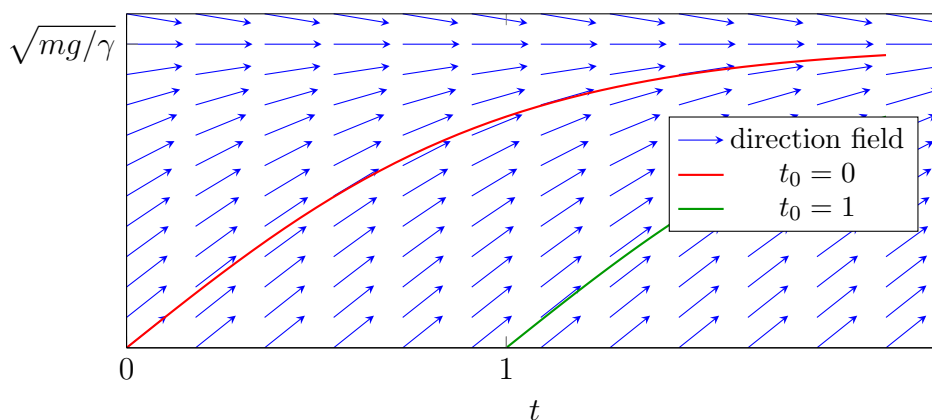
$$\chi(\tau) = \chi(\tau_0) + \log(\cosh(\tau - \tau_0)).$$

Restoring x and t ,

$$\frac{x(t)}{x_c} = \frac{x_0}{x_c} + \log \left[\cosh \left(\frac{t - t_0}{t_c} \right) \right],$$

which is the stated formula.

The equation is readily inverted to give time of flight as a function of position. The figure shows dimensionless velocity $\chi'(\tau) = \tanh(\tau - \tau_0)$ for $\tau_0 = 0$ and $\tau_0 = 1$. Solutions approach the terminal speed $\sqrt{mg/\gamma}$.



9. Let a be a constant and let q be continuous on an interval I . Verify that

$$\phi(t) = x_0 e^{at} + e^{at} \int_{t_0}^t e^{-as} q(s) \, ds \quad (4)$$

solves

$$\begin{cases} x' = ax + q(t), \\ x(t_0) = x_0. \end{cases} \quad (5)$$

Show that any other solution $\psi: J \rightarrow \mathbb{R}$ of (5), defined on an interval $J \subseteq I$ containing t_0 , is a restriction of (4).

Solution. Differentiating (4) by the product rule and the fundamental theorem of calculus gives

$$\phi'(t) = a\phi(t) + q(t), \quad \phi(t_0) = x_0,$$

so ϕ solves the IVP.

If ψ is any other solution, then $x = \psi - \phi$ satisfies

$$x' = ax, \quad x(t_0) = 0.$$

The zero function is a solution. If ξ is any solution on an interval $J \ni t_0$,

$$\frac{d}{dt}(\xi(t)e^{-at}) = (\xi' - a\xi)e^{-at} = 0,$$

so $\xi(t)e^{-at}$ is constant on J . The condition $\xi(t_0) = 0$ forces the constant to be 0, hence $\xi \equiv 0$ on J . Thus $\psi = \phi$ on J .

10. In the interior of a homogeneous spherical planet the gravitational force on a mass m at distance x from the centre is $F(x) = -kmx$, with $k > 0$. A frictionless tunnel is bored from pole to pole and a ball of mass m is placed in the tunnel.
- (a) Write an IVP for release from rest at the centre $x = 0$ at time $t = 0$, and show that the unique solution is the zero function on $\mathbb{R}$.
 - (b) Find the position as a function of time if the ball is dropped from a pole at $t = 0$. Find also the velocity at the centre and the time from pole to centre.
 - (c) Straight tunnels are drilled from London to Moscow and from London to New York. Capsules are released simultaneously from London in each tunnel and travel under gravity alone. Show that they arrive at their destinations, and at the same time.

Solution. Newton's law gives $x'' + kx = 0$.

- (a) The IVP is

$$x'' + kx = 0, \quad x(0) = 0, \quad x'(0) = 0.$$

Multiplying by x' yields

$$\frac{d}{dt} \left(\frac{(x')^2}{2} + k \frac{x^2}{2} \right) = 0,$$

so $(x')^2 + kx^2$ is constant. The initial conditions force $(x')^2 + kx^2 = 0$. Since $k > 0$, this implies $x(t) = x'(t) = 0$ for all t . The ball remains at the centre.

- (b) Now $x(0) = R$ and $x'(0) = 0$, where R is the planetary radius. The solution is

$$x(t) = R \cos(\sqrt{k}t).$$

Hence

$$x'(t) = -R\sqrt{k} \sin(\sqrt{k}t).$$

The centre $x = 0$ is reached at $t = \pi/(2\sqrt{k})$, with velocity $-R\sqrt{k}$. (The same speed follows from conservation of energy.)

- (c) Let ξ be displacement along a chord of length L , measured from the midpoint of the tunnel, and let x be distance from the Earth's centre. Then $\xi = x \cos \theta$, where θ is the angle between the radius to the mass and the tunnel. The component of force along the tunnel is

$$F \cos \theta = -kmx \cos \theta = -km\xi,$$

so

$$\xi'' + k\xi = 0.$$

With $\xi(0) = L/2$ and $\xi'(0) = 0$,

$$\xi(t) = \frac{L}{2} \cos(\sqrt{k} t).$$

The period $2\pi/\sqrt{k}$ is independent of L . A mass starting at $\xi = L/2$ with zero velocity reaches $\xi = -L/2$ in half a period, i.e. in time $\pi/\sqrt{k}$, the same for every chord.