Extension of Induction to Well Founded Sets[1]

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1. A ordered set X is called *well-founded* (abbreviated WF) if every non empty subset of X has a minimal element.

2. A set is *well ordered* if it is:

   (a) Well founded and

   (b) Totally ordered

3. Zermelo's theorem (equivalent to the axiom of choice).

   Every set can be well-ordered.

4. Special Case: Well-ordering principle:

   Every non-empty set of the natural numbers contains a least element.

5. $(X, \leq)$ be an order. The following are equivalent:

   (a) X is well founded.

   (b) There is no strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$ of elements of X with an infinite number of terms.

   Proof. Let X be well founded. A strictly decreasing sequence with infinite image can have no minimal element. It follows that no such sequence can exist in X . Suppose now that there is no strictly decreasing sequence

   $$x_0 > x_1 > x_2 > \cdots$$

   of elements of X with an infinite number of terms and to the contrary, there exists a non empty subset B with no minimal element. Let $b_0 \in B$. Since b_0 is not minimal there exists $b_1 \in B$ such that $b_1 > b_0$. Continuing in like manner we construct a strictly decreasing sequence with an infinite image - contradiction.

6. $(X, \leq)$ be an order. The following are equivalent:

   (a) X is well founded.

   (b) Strong Induction (SI):

   Let $p : X \to \{T, F\}$ be a predicate. Suppose that for all elements $x \in X$ the statement

   $$[\forall y < x, \quad p(y)] \to p(x) \qquad (1)$$

   is true. Then the property p is true for all $x \in X$.

   Proof. $(\Rightarrow)$ Let X be well founded. Suppose to the contrary that induction does not hold. Then there exists a property $p : X \to \{T, F\}$ and an element $x^* \in X$ such that $p(x^*)$ is false yet for all x, (1) is true. By WF we may assume x^* is minimal such that $p(x^*)$ is false. Then $[y < x^*, p(y)]$ is true and it follows by (1) that $p(x^*)$ is also true – a contradiction.

   $(\Leftarrow)$ Suppose that strong induction holds. We show that any set $A \subseteq X$ which has no minimal element must be empty. Consider the property $y \notin A$. If A has no minimal element the statement

   $$[\forall y < x, y \notin A] \to x \notin A$$

   must be true for all x. (If there existed $x \in A$ such that $[\forall y < x, y \notin A]$) then x would be minimal in A. It follows by induction that A is empty as advertised.

7. Remark: If x^* is minimal in X, then the left hand side of (1) is vacuously true. Hence the basis of induction is vacuously included in the statement of SI.

[1]Reference: Wikipedia article *Well-founded relation*

[2] http://pennance.us