Relations in Graphic Matroids[1]

1. Definition.

   Let E be a set and let $\mathcal{A} \subseteq 2^E$. The following notation will be used for certain subsets of 2^E

   (a) $\neg\mathcal{A} = \{X \in 2^E : X \notin \mathcal{A}\}$.

   (b) $\perp\mathcal{A} = \{X \in 2^E : |X \cap A| \neq 1,\ \forall A \in \mathcal{A}\}$.

   (c) $\mathbf{cs}\,\mathcal{A} = \{X \in 2^E : X \text{ contains some } A \in \mathcal{A}\}$.

   (d) $\mathbf{cn}\,\mathcal{A} = \{X \in 2^E : X \text{ contains no } A \in \mathcal{A}\}$.

   (e) $\mathbf{cis}\,\mathcal{A} = \{X \subseteq E : X \text{ is contained in some } A \in \mathcal{A}\}$.

   (f) $\mathbf{cin}\,\mathcal{A} = \{X \subseteq E : X \text{ is contained in no } A \in \mathcal{A}\}$.

   (g) $\mathbf{Max}\,\mathcal{A} = \{A \in \mathcal{A} : A \text{ is maximal in } \mathcal{A}\ \}$.

   (h) $\mathbf{Min}\,\mathcal{A} = \{A \in \mathcal{A} : A \text{ is minimal in } \mathcal{A}\ \}$.

   (i) $\mathbf{Comp}\,\mathcal{A} = \{X \subseteq E : E \setminus X \in \mathcal{A}\}$.

2. Claim.

   Let G be an e-labelled graph with edge set $E = E(G)$. Any of the following lists describes the matroid $M(G)$ of G.

   (a) $\mathcal{C}(G)$ – cycles of G.

   (b) $\mathcal{C}^*(G)$– cocycles (minimal cuts).

   (c) $\mathcal{T}(G)$ – frames (maximal forests).

   (d) $\mathcal{T}^*(G)$ – cotrees of G.

   (e) $\mathcal{F}(G)$ – forests (acyclic).

   (f) $\mathcal{F}^*(G)$ – co-forests (a-cocyclic).

3. Some Relations.

   (a) $\mathcal{F}(G) = \mathbf{cn}\,\mathcal{C}(G)$.

   (b) $\mathcal{T}(G) = \mathbf{Max}(\mathcal{F})$.

   (c) $\mathcal{F}(G) = \mathbf{cis}\,\mathcal{T}(G)$.

   (d) $\mathcal{C}(G) = \mathbf{Min\,cin}\,\mathcal{T}(G)$.

   (e) $\mathcal{T}(G) = \mathbf{Comp}\,\mathcal{T}^*G)$.

   (f) $\mathcal{F}^*(G) = \mathbf{cn}\,\mathcal{C}^*(G)$.

   (g) $\mathcal{T}^*(G) = \mathbf{Max}\,\mathcal{F}^*(G)$.

   (h) $\mathcal{F}^*(G) = \mathbf{cis}\,\mathcal{T}^*(G)$.

   (i) $\mathcal{C}^*(G) = \mathbf{Min\,cin}\,\mathcal{T}^*(G)$.

   (j) $\mathcal{T}^*(G) = \mathbf{Comp}\,\mathcal{T}(G)$

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\begin{tikzcd}
\mathcal{C}^*(G) \arrow[rrr, "\mathbf{Min}\perp"] \arrow[dd, "\mathbf{cn}"'] & & & \mathcal{C}(G) \arrow[dl, "\mathbf{cn}"'] \\
& \mathcal{T}^*(G) \arrow[ul, "\mathbf{Min\,cin}"'] \arrow[r, leftrightarrow, "\mathbf{Comp}"] & \mathcal{T}(G) \arrow[dr, "\mathbf{cis}"] & \\
\mathcal{F}^*(G) \arrow[ur, "\mathbf{Max}"] \arrow[rrr] & & & \mathcal{F}(G) \arrow[uu, "\mathbf{cin}"']
\end{tikzcd}
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[1] Notes by Philip Pennance – based on Lectures by Alexander Kelmans at the University of Puerto Rico, cerca 1996.