Inverse Functions

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1. Claim.

   Let $f : A \to B$. The following are equivalent:

   (a) f is surjective.

   (b) f has a right inverse.

   Proof.

   (a) $\Rightarrow$ (b): Suppose that f is surjective. Let
   $$\mathcal{F} = \{f^{-1}(b) : b \in B\}$$
   By the axiom of choice there exists a function
   $$\phi : \mathcal{F} \to \bigcup \mathcal{F}$$
   such that $\phi(M) \in M$ for all $M \in \mathcal{F}$. Since f is surjective, $\bigcup \mathcal{F} = A$. Define $g : B \to A$ by
   $$g(b) = \phi(f^{-1}(b))$$
   then $f \circ g(b) = b$ for all $b \in B$, i.e., $g : B \to A$ is a right inverse of f.

   (b) $\Rightarrow$ (a): Suppose f has a right inverse g. Then, for all $b \in B$
   $$\begin{aligned} b &= \mathrm{id}_B(b) \\ &= f \circ g(b) \end{aligned}$$
   and so f is surjective.

2. Exercise. Use the method suggested by the proof to find a right inverse of
   $$f : \{-2, -1, 0, 1, 2\} \to \{0, 1, 4\}$$
   given by $f(x) = x^2$. How many right inverses are there?

3. Claim.

   Let $f : A \to B$. The following are equivalent:

   (a) f is injective.

   (b) f has a left inverse.

   Proof. Exercise.

4. Exercise. Let $f : [0, 1] \to \mathbb{R}$ be given by
   $$f(x) = 2x + 1.$$
   Find two left inverses of f.

5. Let $f : A \to B$ be a bijection. If f has a left inverse g_L and a right inverse g_R, and an inverse g, then
   $$g_l = g_R = g$$
   Proof.

$$\begin{tikzcd}
B \arrow[loop left, "\mathrm{id}_B"] \arrow[r, shift right, "g_R"'] & A \arrow[l, shift right, "f"'] \arrow[r, shift left, "f"] \arrow[d, shift right, "f"'] \arrow[loop above, "\mathrm{id}_A"] & B \arrow[l, shift left, "g_R"] \arrow[loop right, "\mathrm{id}_B"] \\
 & B \arrow[u, shift right, "g"'] \arrow[loop below, "\mathrm{id}_B"] &
\end{tikzcd}$$

$$\begin{aligned} g &= g \circ \mathrm{id}_B \\ &= g \circ (f \circ g_R) \\ &= (g \circ f) \circ g_R \\ &= \mathrm{id}_A \circ g_R \\ &= g_R \end{aligned}$$

$$\begin{aligned} g &= \mathrm{id}_A \circ g \\ &= (g_L \circ f) \circ g \\ &= g_L \circ (f \circ g) \\ &= g_L \circ \mathrm{id}_B \\ &= g_L \end{aligned}$$

[1] http://pennance.us