

Equivalence Relations

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1. Definition.

Let V be a set. A binary relation $E \subseteq V \times V$ is an *equivalence relation* if it is reflexive, transitive and symmetric.

2. Definition.

Let E be an equivalence relation on V . The set

$$[v] = \{x \in V : (v, x) \in E\}$$

is called the *equivalence class* of v .

3. Notation.

If $E \subseteq V \times V$ is an equivalence relation, we will write $a \sim b$ instead of $(a, b) \in E$.

4. Examples of Equivalence Relations:

(a) *Similarity* and *congruency* on the set of all triangles.

(b) *Congruency modulo n* on $\mathbb{Z}$.

$$x \sim y \Leftrightarrow n \mid (y - x).$$

(c) The relation

$$x \sim x' \Leftrightarrow f(x) = f(x')$$

on the domain of a function f .

5. Exercise.

Let E be an equivalence relation on V . Let $a, b \in V$. The following are equivalent:

(a) $[a] = [b]$

(b) $a \in [b]$

6. Claim.

Let E be an equivalence relation on V . Let $a, b \in V$. The following are equivalent:

$$[a] \neq [b] \tag{1}$$

$$[a] \cap [b] = \emptyset \tag{2}$$

Proof.

(1) $\Rightarrow$ (2). Suppose $[a] \neq [b]$ and, to the contrary, that (2) is false. Then $\exists x \in [a] \cap [b]$. By the previous exercise, $[x] = [a]$ and $[x] = [b]$, and so $[a] = [b]$ – a contradiction.

(2) $\Rightarrow$ (1). Suppose $[a] \cap [b] = \emptyset$. If, to the contrary $[a] = [b]$ then, by the previous exercise $a \in [b]$.

7. Corollary.

Let E be an equivalence relation on V . Then the set of equivalence classes of E is a partition of V .

8. Definition.

Let V be a set and $E \subseteq V \times V$. A subset $C \subseteq V$ is called a *clique* if for all $x, y \in V$

$$x, y \in C \Rightarrow (x, y) \in E$$

A clique is *maximal* if it is not properly contained in another clique.

9. Exercise.

Let E be an equivalence relation on V . Show that the following are equivalent:

(a) C is a maximal clique.

(b) $C = [c]$ for some $c \in C$.

¹<https://pennance.us>