

# MATH 3225 Homework on Induction

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1. Prove for all  $n \in \mathbb{N}$ :

(a)  $\sum_{i=1}^n (2i-1)^2 = n^2$

(b)  $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$

(c)  $\sum_{i=1}^n (-1)^i = \frac{(-1)^n - 1}{2}$

(d)  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

2. Let  $x \in \mathbb{R}, x \neq 1$ . Show that for all  $n \in \mathbb{N}$

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}$$

3. Prove:

(a)  $2n \leq 2^n$  for all  $n \in \mathbb{N}$

(b)  $3^n \geq 2n + 1$  for all  $n \in \mathbb{N}$  with  $n \geq 1$

4. Prove that the following expressions are divisible by 3 for any natural number  $n > 1$ .

(a)  $n^3 + 2n$

(b)  $2^{2n} - 1$

5. If  $n \in \mathbb{N}$  show that 6 is a factor of  $2^n + 3^n - 5^n$

6. Show by induction that  $\frac{d}{dx}x^n = nx^{n-1}$  for  $n \in \mathbb{N}$ .

7. Explain carefully what is wrong with the following induction argument. In any set of  $n$  horses all horses have the same color. Proof: The statement is clearly true if  $n = 1$ . Suppose it is true for  $k$ . Let  $S$  be a set of  $k + 1$  horses.

Delete a horse  $a$  from  $S$ . The set  $S \setminus \{a\}$  has exactly  $k$  horses so by the induction hypothesis all have the same color. Suppose this color is red. Return the horse  $a$  to the set and delete another horse  $b$ .  $S \setminus \{b\}$  has  $k$  elements so every horse in  $S \setminus \{b\}$  must have the same color. Thus  $a$  must have the same color as the other horses in  $S \setminus \{b\}$  namely red. It follows that every horse in  $S$  is red. By induction In any set of  $n$  horses, all horses have the same color.

8. Prove that every natural number is either odd or even.

9. Show that any natural number  $n \geq 2$  is either prime or a product of primes.

10. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $g : \mathbb{N} \rightarrow \mathbb{N}$  be two sequences. Suppose:

(a)  $f(1) = 1$

(b)  $g(1) = 1$

(c)  $f(n+1) = (n+1)f(n), \quad \forall n \in \mathbb{N}$

(d)  $g(n+1) = (n+1)g(n), \quad \forall n \in \mathbb{N}$

Show that  $f = g$ .

11. A sequence  $(a_n)_{n=0}^\infty$  is defined recursively by

(a)  $a_1 = 3$

(b)  $a_k = 4a_{k-1}$  for all  $k \geq 2$ .

Show that  $a_n = 3 \cdot 4^{n-1}$ , for all  $n \in \mathbb{N}$

12. Let  $(a_n)_{n=1}^\infty$  be a sequence of real numbers and  $c \in \mathbb{R}$ . Show that

$$\sum_{i=1}^n (ca_i) = c \sum_{i=1}^n a_i$$

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13. (a) Let  $(a_n)_{n=0}^{\infty}$  be a sequence of real numbers. Prove:

$$a_n = a_0 + \sum_{i=1}^n (a_i - a_{i-1})$$

for all  $n \in \mathbb{N}$ ,

- (b) By taking  $a_i = i^2$  in part (a) find a formula for  $1 + 2 + 3 + \cdots + n$ .

- (c) Use the formula in part (a) to find formula for  $\sum_{k=1}^n k^2$ .

14. A function  $F(n)$  is defined recursively by

(a)  $F(0) = 1; \quad F(1) = 1$

(b)  $F(n) = F(n-1) + F(n-2); n \geq 1$ .

If

$$\alpha = \frac{1 + \sqrt{5}}{2}; \quad \beta = \frac{1 - \sqrt{5}}{2}$$

Use strong induction to show that

$$F(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

15. Prove by induction that  $n$  cents can be obtained using 2 cent and 5 cent coins provided that  $n \geq 4$ .

16. Show that a set of  $n$  elements has exactly  $2^n$  subsets.

17. A plane is divided into regions by a finite number of lines. Show that each region can be colored by one of two colors in such a way that regions sharing a side have different colors.

18. Prove Bernoulli's inequality

$$(1 + x)^n \geq 1 + nx$$

for all non negative integers  $n$  and real numbers  $x > -1$