Math 3325 - Some Axioms for Set Theory

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It is a never ending debate as to which axioms for set theory are the best/most useful/plausible/self evident/etc./etc., in one or other specified sense. This is a highly technical question which will not be entered into. The following is a partial list of of the axioms Zermelo and Fraenkel which in one way or another will be used implicitly or explicitly in this course. Each axiom is stated formally in the language of ZF set theory as well as informally in the manner used by most mathematicians and non specialists. More information can be found in https://en.wikipedia.org/wiki/ZermeloFraenkel_set_theory

1. *Axiom of Extension*

   $$\forall x \forall y[(x = y \Leftrightarrow \forall z \quad (z \in x \leftrightarrow z \in y)].$$

   Two sets are *equal* if and only if they have the same elements.

2. *Existence of the Empty Set*

   $$\exists x \forall y \quad \neg y \in x.$$

   There exists an empty set.

3. *Axiom of Specification*

   $$\forall y \exists z \forall x (x \in z \leftrightarrow (x \in y \wedge p(x)).$$

   If y is a set and $p(x)$ a predicate with free variable x, then

   $$z = \{x \in y : p(x)\}$$

   is a set.

4. *Existence of Unordered Pairs*

   $$\forall x \forall y \exists z (w \in z \leftrightarrow (w = x \vee w = y)).$$

   Given sets x and y there exists a set whose elements are precisely x and y.

5. *Axiom of Unions*

   $$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x)).$$

   If x is a set of sets, then there exists a set $\cup x$ whose elements are precisely the elements of elements of x. For example.

   $$\bigcup \{\{1, 2\}, \{2, 3\}, \{3, 4\}\} = \{\{1, 2, 3, 4\}.$$

6. *Power Set*

   $$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x).$$

   For every set x there is a set y, called the *power set*, whose elements are precisely the subsets of x.

7. *Existence of an Inductive Set*

   $$\exists x (\phi \in x \wedge \forall y (y \in x \to y \bigcup \{y\} \in x)).$$

   There exists a set x such that

   (a) $\phi \in x$ and

   (b) For all $y \in x$ we have $y \bigcup \{y\} \in x$

8. *Axiom of Choice*

   Informal Version: Let x be a set of non empty sets. Then $\exists c : x \to \bigcup x$ such that $c(y) \in y$, $\forall y \in x$.

   The axiom of choice, is needed only for infinite sets. It has many equivalent formulations, among them Zorn's lemma and the well ordering theorem. It is known to be independent of the other ZF axioms.

   As Bertrand Russel noted, the axiom of choice is needed to choose one sock from each of an infinite number of pairs of socks. On the other hand, the axiom is not needed in the case of infinitely many shoes since the left shoe can be chosen from each pair.

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