

Applications of the Gradient

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1. Example.

Find tangent and normal lines to the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

at a point (x_0, y_0) .

Solution.

Let

$$f(x, y) = b^2x^2 - a^2y^2$$

and let $\alpha(t) = (x(t), y(t))$ be a parametrization of the level curve

$$f(x, y) = a^2b^2$$

with $\alpha(t_0) = (x_0, y_0)$. Then.

$$f(\alpha(t)) = a^2b^2.$$

By the chain rule,

$$\nabla f'(\alpha(t)) \cdot \alpha'(t) = 0.$$

It follows that

$$\nabla f(\alpha(t_0)) = (2b^2x_0, -2a^2y_0)$$

is normal to the curve at (x_0, y_0) and that

$$\nabla f^\perp(\alpha(t_0)) = (2a^2y_0, 2b^2x_0)$$

is tangent.

The tangent line at $P = (x_0, y_0)$ is

$$(X - P) \cdot \nabla f(x_0, y_0) = 0$$

$$\iff (x - x_0, y - y_0) \cdot (2b^2x_0, -2a^2y_0) = 0$$

$$\iff (x - x_0)2b^2x_0 - (y - y_0)(2a^2y_0) = 0.$$

The normal line at $P = (x_0, y_0)$

$$(X - P) \cdot \nabla f^\perp(x_0, y_0) = 0$$

$$\iff (x - x_0, y - y_0) \cdot (2a^2y_0, 2b^2x_0) = 0$$

$$\iff (x - x_0)(2a^2y_0) + (y - y_0)(2b^2x_0) = 0.$$

2. Example.

Let

$$f(x, y) = x^2 - y^2.$$

Find a family of curves, orthogonal to the family of hyperbolas

$$f(x, y) = c, \quad c > 0.$$

Since $\nabla f(x, y) = (2x, -2y)$ is normal to a member of the family at (x, y) we seek curves $\beta(t) = (x(t), y(t))$ which are parallel to $\nabla f(x, y)$. Such a curve must satisfy the system of ordinary differential equations.

$$\begin{cases} \frac{dx}{dt} = M(x, y) = 2x \\ \frac{dy}{dt} = N(x, y) = -2y \end{cases} \quad (1)$$

If $x \neq 0$ this determines a non autonomous equation

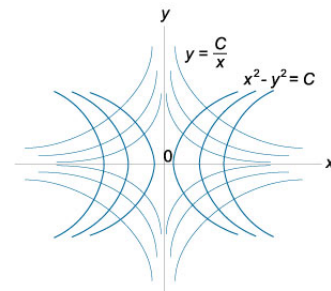
$$\frac{dy}{dx} = \frac{-y}{x}$$

often written

$$-y dx + x dy = 0$$

Remark: The vector field (y, x) not exact. Nevertheless, it can be solved by separation of variables.

$$\begin{aligned} \int_{y_0}^y \frac{1}{y} dy &= - \int_{x_0}^x \frac{1}{x} dx \\ \implies \ln \frac{y}{y_0} &= - \ln \frac{x}{x_0} \\ \implies y &= \frac{x_0 y_0}{x} = \frac{c}{x}. \end{aligned}$$



¹<https://pennance.us>

3. Example.

Let

$$f(x, y) = x^2 + y^2.$$

Find a family of curves, orthogonal to the family of circles

$$f(x, y) = c, \quad c > 0.$$

Since $\nabla f(x, y) = (2x, 2y)$ is normal to a member of the family at (x, y) we seek curves $\beta(t) = (x(t), y(t))$ which are parallel to $\nabla f(x, y)$. Such a curve must satisfy the system of ordinary differential equations.

$$\begin{cases} \frac{dx}{dt} = M(x, y) = 2x \\ \frac{dy}{dt} = N(x, y) = 2y \end{cases} \quad (2)$$

If $x \neq 0$ this determines a non autonomous equation

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} \\ y dx - x dy &= 0 \end{aligned}$$

Remark: The vector field $(y, -x)$ is not exact but can be made exact by multiplying by the integrating factor $\frac{1}{xy}$

yielding

$$\frac{1}{x} dx - \frac{1}{y} dy = 0$$

It is easy to verify that

$$\phi(x, y) = \ln \frac{x}{y}$$

is a potential function. Level curves of the potential function are solutions of the non autonomous equation. (See [here](#) for a proof of this fact)

$$\begin{aligned} \ln \frac{x}{y} &= \text{constant} \\ \implies y &= Cx, \quad x \neq 0 \end{aligned}$$

Hence, the family of curves orthogonal to the family of circles consists of rays located at the origin.

