Iterated Integrals and Charge Density

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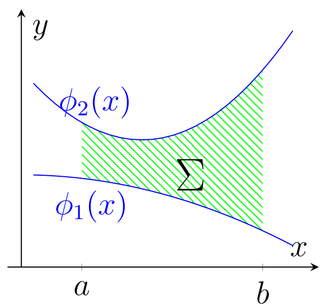
1. Claim.

   Let ϕ_1 and ϕ_2 be continuous functions defined on an interval $[a,b]$ with $\phi_2(x) > \phi_1(x)$ for all $x \in [a,b]$. Let Σ be the region bounded by the the graphs of ϕ_1 and ϕ_2 between $x = a$ and $x = b$.



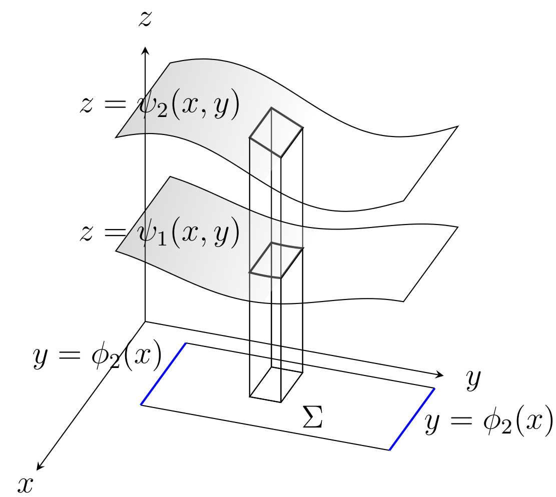
   If $\sigma : \Sigma \to \mathbb{R}$ is a continuous function, then
   $$\int_\Sigma \sigma = \int_a^b \lambda$$
   where
   $$\lambda(x) = \int_{\phi_1(x)}^{\phi_2(x)} \sigma(x,y)dy.$$
   I.e.,
   $$\int_\Sigma \sigma(x,y)\,dxdy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \sigma(x,y)dy\,dx$$

   Motivation.

   Think of $\sigma(x,y)$ as a surface charge density. The first integration reduces this to a line charge density $\lambda(x)$. The integral of the induced density $\lambda(x)$. over the line segment $[a,b]$ yields the total charge. In the case $\sigma(x,y) = 1$ the iterated integral yields the area of the region Σ.

2. Generalization to Three Variables

   Let Σ be the subset of the xy-plane described in the previous claim. Let $z = \psi_2$ and $z = \psi_2$ be two surfaces with
   $$\psi_2(x,y) > \psi_1(x,y$$
   for all $(x,y) \in \Sigma$. Let Ω be the bounded region consisting of all points (x,y,z) with $(x,y) \in \Sigma$ and falling between the two surfaces so that $\psi_1(x,y) < z < \psi_2(x,y)$.



   If $\rho : \Omega \to \mathbb{R}$ is a continuous function, then
   $$\int_\Omega \rho = \int_\Sigma \sigma$$
   where
   $$\sigma(x,y) = \int_{\psi_1(x,y)}^{\psi_2(x,y)} \rho(x,y,z)dz.$$
   Hence, using the previous claim
   $$\int_\Omega \rho = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} \rho(x,y,z)dzdy\,dx$$

   Motivation.

   Think of $\rho(x,y,z)$ as a 3 dimensional volume charge density. The right most integral reduces this to a surface charge density σ . The middle integral then reduces this two dimensional surface density to a one dimensional line charge density, which is then integrated over the line segment $[a.b]$ to produce the total charge. In the case $\rho(x,y,z) = 1$ the triple integral yields the volume of the region Ω.

[1] https://pennance.us