

Math 3153 - Gradient Example

Philip Pennance¹ April 02, 2016

Let $f(x, y) = x^2 + y^2$. Find:

- (a) The linear map $Df(1, 1)$.
- (b) The rate of increase of f in the direction of $(3, 4)$ at the point $(1, 1)$.
- (c) An equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, 1, 2)$.
- (d) A normal vector to the surface

$$z = x^2 + y^2$$

at the point $(1, 1, 2)$.

- (e) A normal vector to the level curve of f at $(1, 1)$.
- (f) A tangent vector to the level curve of f at $(1, 1)$.

Solutions.

(a)

$$\nabla f(x, y) = (2x, 2y).$$

$$\nabla f(1, 1) = (2, 2).$$

$$\begin{aligned} Df(1, 1)(h_1, h_2) &= \nabla f(1, 1) \cdot (h_1, h_2) \\ &= 2h_1 + 2h_2 \end{aligned}$$

- (b) Let $u = (\frac{3}{5}, \frac{4}{5})$. Since u is a unit vector in the direction of $(3, 4)$ the rate of increase of f in that direction is given by the directional derivative $D_u f(P)$.

$$\begin{aligned} D_u f(P) &= Df(1, 1)(3/5, 4/5) \\ &= \frac{14}{5} \end{aligned}$$

- (c) The tangent plane to $z = f(x, y)$ when $(x, y) = (1, 1)$ is given by

$$\begin{aligned} z &= f(1, 1) + Df(1, 1)((x, y) - (1, 1)) \\ &= 2 + (2, 2) \cdot ((x, y) - (1, 1)) \\ &= 2 + 2(x - 1) + 2(y - 1) \\ &= 2x + 2y - 2 \end{aligned}$$

I.e.,

$$2x + 2y - z = 2$$

- (d) Let $g(x, y, z) = x^2 + y^2 - z$. Then

$$\nabla g = (2x, 2y, -1)$$

is a normal to the tangent plane at any point (x, y, z) . Hence

$$\vec{n} = (2, 2, -1)$$

is a normal to the tangent plane at the point $P = (1, 1, 2)$.

Remark. The tangent plane at $P = (1, 1, 2)$ has equation

$$(X - P) \cdot \vec{n} = 0.$$

yielding an alternative solution to part (c)

- (e) The vector $\nabla f(1, 1) = (2, 2)$ is normal to the level curve at $(1, 1)$. Hence, $u^\perp = (-2, 2)$ is tangent at $(1, 1)$.

¹<https://pennance.us>