Reformulations of the Notion of Derivative.

Prof. Philip Pennance[1] -Version: March 1, 2022 .

1. The concept of the derivative can be reformulated in terms of linear functions. We consider first the case of a single variable.

2. Definition.

   A function $f : \mathbb{R} \to \mathbb{R}$ is *linear* if for all $x, x' \in R$ and $c \in \mathbb{R}$

   (a) $f(x + x') = f(x) + f(x')$.

   (b) $f(cx) = cf(x)$.

3. Exercise.

   Show that $L : \mathbb{R} \to \mathbb{R}$ is linear if and only if there exists $c \in \mathbb{R}$ such that $L(x) = mx$ for all $x \in \mathbb{R}$.

4. Definition.

   A function $f : \mathbb{R} \to \mathbb{R}$ is called affine if there exists a constant b and a linear function L such that

   $$f(x) = L(x) + b.$$

5. Claim.

   If f is affine, there exists a linear function $L(h)$ such that

   $$f(x + h) = f(x) + L(h).$$

6. Claim.

   Let $I \subseteq \mathbb{R}$ be an open interval. and $f : I \to \mathbb{R}$. The following are equivalent:

   (a) f is differentiable at $x \in I$.

   (b) There exists a linear function L and a function $\phi(h)$ such that:

      i. $f(x+h) = f(x)+L(h)+\phi(h)$.

      ii. $\lim_{h\to 0} \frac{\phi(h)}{h} = 0$.

   Proof.

   Suppose f is differentiable at $x \in I$. Let L be the linear function given by

   $$L(h) = f'(x)h.$$

   Define, for h small enough,

   $$\phi(h) = f(x + h) - f(x) - L(h)$$

   Since f is differentiable

   $$\lim_{h\to 0} \frac{\phi(h)}{h} = 0.$$

   Conversely, if there exist $\zeta \in \mathbb{R}$ and a function ϕ such that

   $$f(x + h) = f(x) + \zeta h + \phi(h)$$

   where $\lim_{h\to 0} \frac{\phi(h)}{h} = 0$, then

   $$\frac{f(x + h) - f(x)}{h} = \zeta + \frac{\phi(h)}{h}$$

   Taking the limit as h approaches 0 shows that $f'(x)$ exists and equals ζ.

7. The previous claim provides a reformulation of the notion of derivate in terms of linear functions.

8. Definition.

   Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is *differentiable* at $x \in I$ if there exists a linear function L and a function $\phi(h)$ such that:

   (a) $f(x + h) = f(x) + L(h) + \phi(h)$

   (b) $\lim_{h\to 0} \frac{\phi(h)}{h} = 0.$

   The linear function $L(h) = f'(x)h$ is called the *derivative* of f at x and is denoted by $Df(x)(h)$. It follows that $Df(x)(1) = f'(x)$

9. The *tangent line* to f at a point $x = p$ is obtained by translation of the graph of the derivative $Df(p)$ by the vector $(p, f(p))$. (see Figure 1).

[1] https://pennance.us

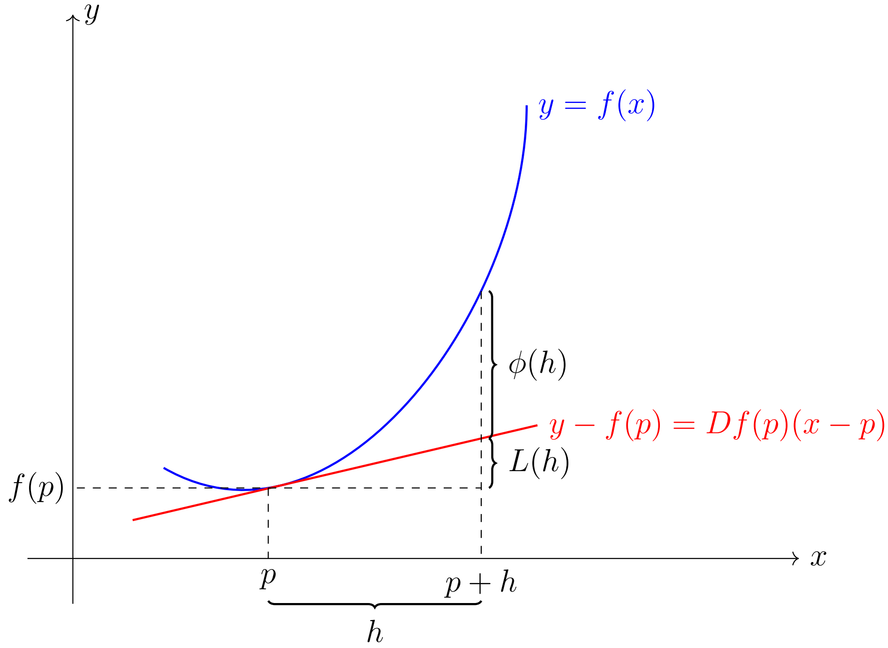

Figure 1. Tangent line at $x = p$.

7. Example.

   If f is linear, then, at any point x,
$$Df(x) = f.$$

8. Example.

   If f is constant, then $Df(x)$ is the zero map:
$$Df(x)(h) = 0 \text{ for all } h \in \mathbb{R}.$$

9. Example.

   If $f(x) = \sin x$, at any point $p \in \mathbb{R}$,
$$Df(p)(h) = (\cos p) \cdot h.$$

   The tangent line at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is given by
$$\begin{aligned} y - \tfrac{1}{2} &= Df\left(\tfrac{\pi}{6}\right)\left(x - \tfrac{1}{2}\right) \\ &= \tfrac{\sqrt{3}}{2}\left(x - \tfrac{1}{2}\right). \end{aligned}$$

Scalar Functions of Several Variables

Prof. Philip Pennance[2] -Version: October 5, 2022

1. Definition.

   Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $\vec{u}$ a unit vector in $\mathbb{R}^n$. The *directional derivative* of f at a point $p \in \Omega$ in the *direction* u is defined by

   $$D_{\vec{u}} f(p) = \lim_{t \to 0} \frac{f(p + t\vec{u}) - f(p)}{t}$$

2. Interpretation. $D_{\vec{u}} f(p)$ is just the rate of change of f in the direction u.

   Remark. If, in the definition, we allow $\|u\| = c$, then $D_{\vec{u}} f(p)$ is the rate of change of f as seen by a point at p moving with speed c in the direction u.

3. Let $f(x, y) = x^2 + y$. Let P be the point $(1, 3)$ and $\vec{u} = (1, 0)$.

   Compute $D_u f(P)$ from the definition.

   Solution.

   $$\begin{aligned} D_{\vec{u}} f(P) &= \lim_{t \to 0} \frac{f(P + t\vec{u}) - f(p)}{t} \\ &= \lim_{t \to 0} \frac{f(1 + t, 3) - f(1, 3)}{t} \\ &= \lim_{t \to 0} \frac{1 + 2t + t^2 + 3 - 4}{t} \\ &= \lim_{t \to 0} \frac{2t + t^2}{t} \\ &= 2. \end{aligned}$$

4. Definition.

   Let $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$ and $\vec{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$. The corresponding directional derivatives:

   $$D_{\vec{e}_1} f(p) = \lim_{t \to 0} \frac{f(p_1 + t, p_2, p_3) - f(p_1, p_2, p_3)}{t}$$

   $$D_{\vec{e}_2} f(p) = \lim_{t \to 0} \frac{f(p_1, p_2 + t, p_3) - f(p_1, p_2, p_3)}{t}$$

   $$D_{\vec{e}_3} f(p) = \lim_{t \to 0} \frac{f(p_1, p_2, p_3 + t) - f(p_1, p_2, p_3)}{t}$$

   are called the *partial derivatives.*

5. Alternative notation:

   $$D_{\vec{e}_1} f(p) = D_1 f(p) = \frac{\partial f}{\partial x}(p) = f_x(p)$$

   $$D_{\vec{e}_2} f(p) = D_2 f(p) = \frac{\partial f}{\partial y}(p) = f_y(p)$$

   $$D_{\vec{e}_3} f(p) = D_3 f(p) = \frac{\partial f}{\partial z}(p) = f_z(p)$$

6. Remark.

   Notice that partial derivatives can be evaluated as ordinary derivatives of a function of a single variable. For example, the student should verify that

   $$D_1 f(p_1, p_2, p_3) = [t \mapsto f(t, p_2, p_3)]'(p_1).$$

7. Example.

   Evaluate $\dfrac{\partial f}{\partial y}(2, 3)$ if

   $$f(x, y) = \frac{|x - 2|}{1 + \sin(y^2 + \tan xy)}.$$

   Solution.

   $$\begin{aligned} D_2 f(2, 3) &= [y \mapsto f(2, y)]'(3) \\ &= [y \mapsto 0]'(3) \\ &= 0 \end{aligned}$$

8. Definition.

   If the partial derivatives exist, the vector $(D_1 f(p), D_2 f(p), \ldots, D_n f)$ is called the *gradient* of f at p and is denoted $\nabla f(p)$. In the case $n = 3$

   $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}. \right)$$

[2] https://pennance.us

9. Definition.

   We say that $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is *differentiable* at $p \in \Omega$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}$ and a function ϕ defined for h small enough so that $p + \vec{h} \in \Omega$, so that

   $$f(p + \vec{h}) = f(p) + L(\vec{h}) + \phi(\vec{h})$$

   and

   $$\lim_{\vec{h}\to 0} \frac{\phi(\vec{h})}{\|\vec{h}\|} = 0 \tag{1}$$

   If f is differentiable at p, the linear map L is written $Df(p)$ and so

   $$f(p + \vec{h}) = f(p) + Df(p)(\vec{h}) + \phi(\vec{h})$$

10. Lemma.

    If $\phi(\vec{h})$ satisfies (1) and k is any unit vector, then

    $$\lim_{t\to 0} \frac{\phi(t\vec{k})}{t} = 0.$$

    Proof.

    Let $\epsilon > 0$. It follows from (1) that there exists $\delta > 0$ such that $\dfrac{\phi(\vec{h})}{\|\vec{h}\|} < \epsilon$ whenever $0 < \|\vec{h}\| < \delta$. In particular, if $h = tk$ where k is a unit vector,

    $$\begin{aligned} 0 < t < \delta &\implies 0 < \|t\vec{k}\| < \delta \\ &\implies \frac{\phi(t\vec{k})}{t} < \epsilon \end{aligned}$$

    and the result follows.

11. Claim.

    If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then the derivative of f is unique.

    Proof.

    If

    $$\begin{aligned} f(x + \vec{h}) &= f(x) + L(\vec{h}) + \phi(\vec{h}) \\ &= f(x) + \hat{L}(\vec{h}) + \hat{\phi}(\vec{h}), \end{aligned}$$

    where L and $\hat{L}$ are linear and ϕ and $\hat{\phi}$ satisfy (1), then, for any h,

    $$L(\vec{h}) - \hat{L}(\vec{h}) = \hat{\phi}(\vec{h}) - \phi(\vec{h}).$$

    Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$, then, for $i = 1, 2$

    $$L(t\vec{e}_i) - \hat{L}(t\vec{e}_i) = \hat{\phi}(t\vec{e}_i) - \phi(t\vec{e}_i),$$

    and by linearity

    $$L(\vec{e}_i) - \hat{L}(\vec{e}_i) = \frac{\hat{\phi}(t\vec{e}_i) - \phi(t\vec{k})}{t}.$$

    It now follows from (1) that

    $$L(\vec{e}_i) = \hat{L}(\vec{e}_i), \quad 1 \le i \le n.$$

    Since the linear maps L and $\hat{L}$ are equal on a basis they must be equal.

12. Examples.

    (a) If f is constant,

    $$Df(p)(\vec{h}) = 0$$

    for all $\vec{h} \in \mathbb{R}^2$.

    (b) If f is linear, then

    $$Df(p) = f$$

    Proof.

    If f is constant, then $f(p + \vec{h}) = f(p)$ and we may take both L and ϕ to be the zero function in the definition of the derivative. By uniquness, this is the only choice.

    If f is linear,

    $$f(a + \vec{h}) = f(a) + L(\vec{h}) + \phi(\vec{h})$$

    reduces to

    $$f(\vec{h}) = L(\vec{h}) + \phi(\vec{h}).$$

    The result follows by taking $L = f$ and $\phi = 0$. Again, this is the only choice.

13. Example.

Let $f(x, y) = x^2 + y^2$. Let $p = (x, y)$ and $\vec{h} = (h, k)$.

$$\begin{aligned} f(p + \vec{h}) &= f(x + h, y + k) \\ &= (x + h)^2 + (y + k)^2 \\ &= f(p) + 2xh + 2yk + h^2 + k^2 \end{aligned}$$

Let

$$\phi(\vec{h}) = h^2 + k^2$$

Since,

$$\lim_{h \to 0} \frac{\phi(\vec{h})}{\|\vec{h}\|} = 0$$

it follows, by uniqness, that the linear term

$$L(\vec{h}) = 2xh + 2yk$$

is the derivative at (x, y). That is,

$$Df(p)(\vec{h}) = 2xh + 2yk.$$

14. Claim.

Let $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. If f is differentiable at p then the partial derivatives $D_i f$ exists at p and the derivative is given by

$$\begin{aligned} Df(p)(\vec{h}) &= \sum_{i=1}^{n} D_i f(p) h_i \\ &= \nabla f(p) \cdot \vec{h}. \end{aligned}$$

Proof (n=2).

Since f is differentiable at p there exist constants a and b such that for h sufficiently small

$$f(p + \vec{h}) = f(p) + ah_1 + bh_2 + \phi(h)$$

Taking $\vec{h} = t\vec{e}_1$ and rearranging yields:

$$\frac{f(p + t\vec{e}_1) - f(p)}{t} = a + \frac{\phi(t\vec{e}_1)}{t}$$

Taking the limit as $t \to 0$ yields $a = D_1 f(p)$. Similarly $b = D_2 f(p)$ and the result is proven.

15. Warning.

The converse of this is false. Existence of $D_1 f(p)$ and $D_2 f(p)$ at a point $p \in \mathbb{R}^2$ does not guarantee differentiability.

16. Claim.

Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $p \in \Omega$. If $D_1 f$ and $D_2 f$ are continous at p, then f is differentiable at p.

Proof.

See S. Lang, *Calculus of Several Variables.*

17. Example.

Find the derivative $Df(p)(\vec{h})$ if $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2x + \sin y$.

Solution.

Let $p = (x, y)$ and $\vec{h} = (h_1, h_2)$

$$\begin{aligned} Df(p)(h_1, h_2) &= \nabla f(x, y) \cdot \vec{h} \\ &= (2, \cos y) \cdot (h_1, h_2) \\ &= 2h_1 + (\cos y)h_2 \end{aligned}$$

18. Remark.

Substituting $X = p + \vec{h}$ in

$$f(p + \vec{h}) = f(p) + Df(p)(\vec{h}) + \phi(\vec{h})$$

it is seen for $\vec{h}$ small that

$$\begin{aligned} f(X) &\approx f(p) + Df(p)(X - p) \\ &= f(p) + \nabla f(p) \cdot (X - p) \end{aligned}$$

If $X = (x, y)$ and $p = (a, b)$ then

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(p)(y - b)$$

We (hopefully) see that near $p = (a, b)$ the graph of f is approximated by the plane

$$z = f(p) + D_1(p)(x - a) + D_2(p)(y - b) \tag{2}$$

19. Definition.

If $z = f(x, y)$ is differentiable at a point $p \in \mathbb{R}^2$, then the subset of $\mathbb{R}^3$ determined by (2) is called the *tangent plane* to the graph of f at a point p.

I.e., the tangent plane at p is translation translation of the graph of the derivative $Df(p)$ by the vector $(p, f(p))$.

20. Example.

Find the tangent plane to the graph of $f(x, y) = 3x+2y$ at the point $p = (1, 1)$.

Solution.

Since $\nabla f(p) = (3, 2)$ and $f(p) = 5$,

$$\begin{aligned} & z - f(p) = Df(p)(X - p) \\ \iff & z - 5 = \nabla f(p) \cdot (x - 1, y - 1) \\ \iff & z - 5 = 3x - 3 + 2y - 2 \\ \iff & z = 3x + 2y \end{aligned}$$

Naturally, the tangent plane to a plane at any point should be the same plane!

21. Claim.

Let f be differentiable at $X \in \mathbb{R}^2$. Then the directional derivative $D_u(f)$ is linear in u, moreover

$$D_u f(X) = Df(X)(u).$$

Proof.

Let $\vec{u}$ be a non zero vector in $\mathbb{R}^2$ Since f is differentiable at X, there exists a function ϕ such that

$$f(X+t\vec{u}) = f(X)+Df(X)(t\vec{u})+\phi(t\vec{u})$$

and

$$\lim_{t\to 0} \frac{\phi(t\vec{u})}{t} = 0.$$

By linearity of the derivative,

$$\frac{\phi(t\vec{u})}{t} = \frac{f(X + t\vec{u}) - f(X)}{t} - Df(X)(\vec{u})$$

and the result follows by taking the limit as $t \to 0$.

22. Corollary.

$$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}.$$

23. Corollary.

$\nabla f(p)$ points in the direction of maximum rate of change of f.

Proof.

By the Cauchy Schwartz inequality

$$|D_u f(p)| \leq \|\nabla f(p)\| \|u\|.$$

If $\vec{u}$ is a unit vector, this implies

$$-\|\nabla f(p)\| \leq D_{\vec{u}} f(p) \leq \|\nabla f(p)\|.$$

Since $\nabla f(P) \cdot \vec{u} = \|\nabla f(P)\| \|\vec{u}\| \cos\theta$ the maximum is achieved when $\vec{u}$ and $\nabla f(p)$ point in the same direction. Moreover,

$$\max\{D_{\vec{u}} f(p) : \|\vec{u}\| = 1\} = \|\nabla f(p)\|.$$

24. Example.

Let $f(x, y) = x^2 + y^2$. Find

(a) The linear map $Df(1, 1)$.

(b) The rate of increase of f in the direction of $(3, 4)$ at the point $(1, 1)$.

(c) An equation of the tangent plane to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

(d) A normal vector to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

Solution.

(a)

$$\begin{aligned} Df(1,1)(h_1, h_2) &= \nabla f(1, 1) \cdot (h_1, h_2) \\ &= 2h_1 + 2h_2. \end{aligned}$$

(b) If u is a unit vector, the rate of increase of f in the direction $[u]$ at a point p is the directional derivative $D_u f(p) = Df(p)(u)$. Taking $u = (\frac{3}{5}, \frac{4}{5})$,

$$\begin{aligned} Df(1,1)(\tfrac{3}{5}, \tfrac{4}{5}) &= 2 \cdot \frac{3}{5} + 2 \cdot \frac{4}{5} \\ &= \frac{14}{5}. \end{aligned}$$

(c) The tangent plane to $z = f(x, y)$ when $p = (1, 1)$ is given by

$$\begin{aligned} z &= f(1,1) + Df(1,1)((x,y) - (1,1)) \\ &= 2 + 2(x-1) + 2(y-1) \end{aligned}$$

Hence the tangent plane at $(1, 1)$ is:

$$2x + 2y + (-1)z + 2 = 0.$$

(d) A normal n to the surface at $(1, 1, 2)$ is just a normal to the tangent plane at that point. Thus $n = (2, 2, -1)$ is a normal vector.

Vector Valued Functions of Several Variables.

Prof. Philip Pennance[3] -Version: March 1, 2022

1. Introduction.

   The notion of the derivative as a linear map extends to Functions of n variables with codomain $\mathbb{R}^m$. This includes the important cases of curves $n = 1$ and vector fields $n = m = 2$ or $n = m = 3$.

2. Definition.

   Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^3$ is *differentiable* at $p \in \Omega$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ and a function Φ such that for $h \in \mathbb{R}^2$ with $\|h\|$ sufficiently small,

   $$f(p + h) = f(p) + L(h) + \Phi(h)$$

   and

   $$\lim_{h\to 0} \frac{\Phi(h)}{\|h\|} = 0.$$

   If f is differentiable at p, the linear map L is written $Df(p)$ and so

   $$f(p + h) = f(p) + Df(p)(h) + \Phi(h).$$

3. Lemma.

   Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear, then the component functions $L_1, L_2, \ldots, L_m$ are linear.

   Proof.

   We consider the special case of a linear map $L : \mathbb{R}^2 \to R^3$.

   $$L(x, y) = (L_1(x, y), L_2(x, y), L_3(x, y))$$

   The general case is similar.

   Let

   $$\pi_i : \mathbb{R}^3 \to \mathbb{R}, \quad 1 \leq 3$$

   be the *projection maps,* defined by

   $$\begin{aligned} \pi_1(x, y, z) &= x \\ \pi_2(x, y, z) &= y \\ \pi_3(x, y, z) &= z. \end{aligned}$$

   Observe that these maps are linear. If $p \in \mathbb{R}^2$ then,

   $$\begin{aligned} L_1(p) &= \pi_1 \circ L(p) \\ L_2(p) &= \pi_2 \circ L(p) \\ L_3(p) &= \pi_3 \circ L(p). \end{aligned}$$

   Since a composition of linear maps is linear, the L_i are linear.

4. Claim.

   Let Ω be an open subset of $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}^3$. If $f = (f_1, f_2, f_3)$ is differentiable at $p \in \Omega$, then the component functions f_i are differentiable at p and the derivative $Df(p) : \mathbb{R}^2 \to \mathbb{R}^3$ has components

   $$Df(p) = (Df_1(p), Df_2(p), Df_3(p)).$$

   Proof.

   Since f is differentiable at p, there exists a function Φ such that for $h \in \mathbb{R}^2$ with $\|h\|$ sufficiently small,

   $$f(p + h) = f(p) + Df(p)(h) + \Phi(h),$$

   and $\lim_{h\to 0} \frac{\Phi(h)}{\|h\|} = 0$. The i-th component is

   $$f_i(p + h) = f_i(p) + Df(p)_i(h) + \Phi_i(h).$$

   By the lemma, each of the $Df(p)_i$ are linear. Moreover, $\displaystyle\lim_{h\to 0} \frac{\Phi_i(h)}{\|h\|} = 0$. By uniqueness of the derivative, it follows that the f_i are differentiable at p and $Df(p)_i = Df_i(p)$.

5. Example.

   Let I be an open interval. If $\alpha : I \to \mathbb{R}^3$ is a differentiable curve, then for each

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$t \in I$, the derivative $D\alpha(t) : \mathbb{R} \to \mathbb{R}^3$ is the linear map given by

$$D\alpha(t) = (D\alpha_1(t), D\alpha_2(t), D\alpha_3(t))$$

where α_1, α_2, α_3 are the component functions of α. Since the α_i are functions from I to $\mathbb{R}$

$$\begin{aligned} D\alpha_1(t)(h) &= \alpha_1'(t)h \\ D\alpha_2(t)(h) &= \alpha_2'(t)h \\ D\alpha_3(t)(h) &= \alpha_3'(t)h \end{aligned}$$

Hence for $h \in R$

$$D\alpha(t)(h) = h(\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)) \quad (3)$$

and, in particular

$$D\alpha(t)(1) = \alpha'(t).$$

6. Example.

Let $\alpha : (0, 2\pi) \to \mathbb{R}^3$ be given by

$$\alpha(t) = (\sin 2t, 2\cos 2t, 3t)$$

Then, for each $t \in (0, 2\pi)$,

$$D\alpha(t) : \mathbb{R} \to \mathbb{R}^3$$

is the linear map given by

$$D\alpha(t)(h) = (2h \cos 2t, -4h \sin 2t, 3h).$$

7. Lemma.

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map with matrix A relative to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Let A_i denote the i-th row of A. Then the i-th component L_i has matrix A_i.

Proof.

Since $L_i = \pi_i \circ L$, the matrix of L_i is the product of the matrix of π_i with the matrix of L. But the matrix of π_i is I_i, the i-th row of the 3×3 identity matrix. Hence,

$$\begin{aligned} \text{Matrix of } L_i &= I_i A \\ &= (IA)_i \\ &= A_i. \end{aligned}$$

8. Corollary.

Let $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be differentiable. The derivative $Df(p)$ has matrix:

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y}. \end{pmatrix}.$$

Notice that the i-th row of J is just the gradient of the component function f_i.

9. Definition.

The matrix J of $DF(p)$ is called the *Jacobian*. If J is a square matrix its determinant is called the *Jacobian determinant*.

10. Theorem. (The Chain Rule for Vector Valued Functions.)

Let U, V be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $g : U \to \mathbb{R}^m$ and $f : V \to \mathbb{R}^p$ with $\operatorname{Im} g \subseteq U$. Let $x \in U$. If $Df(g(x))$ and $Dg(x)$ both exist, then $D(f \circ g)(x)$ exists and

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x). \quad (4)$$

Remarks.

The chain rule shows that the derivative of a composition is the composition of the derivatives. The proof, linked here will be omitted.

11. Important Special Case.

Let $\Omega \subseteq \mathbb{R}^3$ be open and $\alpha : I \to \Omega$ a differentiable curve in Ω. If $f : \Omega \to \mathbb{R}$ is differentiable in Ω then

$$(f \circ \alpha)'(t) = \nabla f(\alpha(t) \cdot \alpha'(t). \quad (5)$$

Proof.

Recall that the derivative of a scalar valued function f is given by

$$Df(p)(\vec{h}) = \nabla f(p) \cdot \vec{h}$$

Hence, by the chain rule and (3)

$$\begin{aligned} D(f \circ \alpha)(t)(h) &= Df(\alpha(t) \circ D\alpha(t)(h) \\ &= \nabla f(\alpha(t) \cdot \alpha'(t)h. \end{aligned}$$

The result follows by putting $h = 1$.

Remark. See here for a direct proof.

12. Alternative notation.

If $\alpha(t) = (x(t), y(t), z(t))$, the form of the chain rule (5) is commonly written:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dt}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

13. Example.

Let
$$f(x, y) = x^2 + y^2$$
and
$$\alpha(t) = \cos t, \sin t).$$
Then

$$\begin{aligned} (f \circ \alpha)'(t) &= D(f \circ \alpha)(t)(1) \\ &= Df(\alpha(t)) \circ D\alpha(t)(1) \\ &= Df(\alpha(t))(\alpha'(t)) \\ &= \nabla f(\alpha(t)) \cdot \alpha'(t) \\ &= (2\cos t, 2\sin t) \cdot (-\sin t, \cos t) \\ &= 0. \end{aligned}$$

(Unsurprisingly).

14. Example.

Let Φ be given as a composition of differentiable functions given by

$$\Phi(r, \theta) = \phi(x(r, \theta), y(r, \theta)).$$

Show that

$$\frac{\partial \Phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r}.$$

Solution.

Notice that $D_1\Phi$ is the ordinary derivative of the map $\alpha : r \to \Phi(r, \theta)$. Since α is a curve,

$$\alpha'(r) = \nabla \phi(\alpha(r)) \cdot \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}\right).$$

and the result follows.

15. Example.

Use the chain rule to show that

$$\nabla \frac{1}{\|X\|} = -\frac{X}{\|X\|^3}. \qquad (6)$$

Solution.

Note that if f is a scalar values function of one variable and $\phi : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ it follows from the chain rule that

$$\nabla(f \circ \phi)(X) = f'(\phi(X))\nabla\phi(X)$$

Taking $f(x) = 1/x$ and $\phi(x) = \|X\|$ yields

$$\begin{aligned} \nabla \frac{1}{\|X\|} &= \nabla f(\|X\|) \\ &= f'(\|X\|)\nabla\|X\| \\ &= -\frac{1}{\|X\|^2} \cdot \frac{X}{\|X\|}. \end{aligned}$$

Remark for students of physics.

Since an inverse square force field has the form $f(X) = -\frac{X}{\|X\|^3}$ It follows that $-\phi(X) = -\frac{1}{\|X\|}$ is a potential energy.

16. Claim.

If

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$$

is a smooth surface, then $\nabla g(p)$ is perpendicular to S at a point p.

Proof.

Let α be any curve in S such that $\alpha(t_0) = p$. Then

$$\begin{aligned} & g(\alpha(t)) = 0 \\ \implies & \nabla g(\alpha(t_0)) \cdot \alpha'(t_0) = 0 \\ \implies & \nabla g(\alpha(t_0)) \perp \alpha'(t_0). \end{aligned}$$

Thus $\nabla g(p)$ is perpendicular to the tangent vector α'. Since α is an arbitrary curve $\nabla g(p)$ is perpendicular to the S at the point p.

17. Remark.

    The same argument shows that ∇f is perpendicular to curves of the form $f(x, y) = c$.

18. Example.

    Let S be the surface given by $z = x^2 + y^2$. Find:

    (a) A normal vector to S at the point $p = (1, 1, 2)$.

    (b) Find an equation of the tangent plane to S at $p = (1, 1, 2)$.

    Solution.

    Let $g(x, y, z) = x^2 + y^2 - z$. Then,

    $$\begin{aligned} \nabla g &= (2x, 2y, -1) \\ \nabla g(1, 1, 2) &= (2, 2, -1). \end{aligned}$$

    Hence $n = (2, 2, -1)$ is a normal to the plane at the point p. Let $X = (x, y, z)$ be any point on the plane with normal vector n incident to p.

    $$\begin{aligned} & (X - p) \cdot n = 0 \\ \iff & 2(x - 1) + 2(y - 1) - (z - 2) = 0 \\ \iff & 2x + 2y - z + 2 = 0. \end{aligned}$$

19. Example.

    Let $f(x, y) = x^2 + y^2$. Find: a normal vector and a tangent vector to the level curve of f at the point $p = (1, 3)$.

    Solution.

    $$\begin{aligned} \nabla f &= (2x, 2y). \\ \therefore \nabla f(p) &= (2, 6) \end{aligned}$$

    Since $u = \nabla f(p) = (2, 6)$ is normal to the level curve at p, it follows that $u^\perp = (-6, 2)$ is tangent.

20. Application. (Conservation of Energy).

    Let I be an interval and $\alpha : I \to \mathbb{R}^3$ the path of a point of mass m subject to a force field $f(x, y, z)$. According to an axiom of mechanics, (written first as a differential equation by L. Euler,

    $$m\alpha''(t) = f(\alpha(t)). \tag{7}$$

    Claim.

    If there exists a scalar function ϕ such that $f = -\nabla\phi$, then

    $$\tfrac{1}{2} m \|\alpha'\|^2 + \phi(\alpha(t) \tag{8}$$

    is constant on I.

    Proof.

    Substituting $f = -\nabla\phi$ in (7)

    $$m\alpha''(t) + \nabla\phi = 0.$$

    Taking the scalar product with $\alpha'(t)$ yields

    $$m\alpha'(t) \cdot \alpha''(t) + \nabla\phi(\alpha(t)) \cdot \alpha'(t) = 0$$

    Using the chain rule, this can be written:

    $$\frac{d}{dt}\left[\frac{m}{2}\alpha'(t) \cdot \alpha'(t) + \phi(\alpha(t)\right] = 0.$$

    Hence, there exists a constant E such that

    $$\tfrac{1}{2} m \|\alpha'\|^2 + \phi(\alpha(t) = E.$$

    Remarks. The quantity $\frac{1}{2} m \|\alpha'\|^2$ is called the *kinetic energy.* The scalar function ϕ (which is unique up to a constant) called the *potential energy.* The constsnt E is called the *total energy.*

Math 3153 - Exercises on the Derivative

Philip Pennance[4] – Version: March 1, 2022

1. Sketch level curves for the following functions:

   (a) $f(x,y) = x^2 + 2y^2$

   (b) $f(x,y) = y - x^2$

   (c) $f(x,y) = 3x^2 + 3y^2$

   (d) $f(x,y) = xy$

2. Find the partial derivatives D_1, D_2, and D_3 for the functions:

   (a) $f(x,y,z) = x^{yz}$

   (b) $f(x,y,z) = \dfrac{x^2 + y^4}{z^2}$

   (c) $f(x,y,z) = \arcsin xy^2z^3$

   (d) $f(x,y,z) = \sin(\sin(\sin xyz)))$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. If $g(r,\theta) = f(r\cos\theta, r\sin\theta)$, calculate $\dfrac{\partial g}{\partial r}$ y $\dfrac{\partial g}{\partial \theta}$ in terms of $\dfrac{\partial f}{\partial x}$ y $\dfrac{\partial f}{\partial y}$.

4. Evaluate:

   (a) $\dfrac{\partial f}{\partial y}(2,3)$ if $f(x,y) = \dfrac{\sin(x-2)}{1 + \sin(y^2 + \tan xy)}$

   (b) $\dfrac{\partial f}{\partial x}(0,0)$ if $f(x,y) = \sqrt{xy}$

5. Determine if the function $v(x,y) = \ln(x^2 + y^2)$, $(x,y) \neq (0,0)$, satisfies the equation:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

6. Let $f(x,y) = x^2 + y$. Let p be the point $(1,3)$ and $\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

   (a) Find the gradient $\nabla f(p)$.

   (b) Use the result of part (a) to find the directional derivative $D_{\vec{u}} f(p)$.

   (c) Compute $D_{\vec{u}} f(p)$ in (b) directly from the definition.

   (d) In which direction is the directional derivative $D_{\vec{u}} f(p)$ maximal? What is the directional derivative in that direction?; in a perpendicular direction?

7. Let $\alpha : (-1,1) \to \mathbb{R}^2$ differentiable with $\alpha'(0) \neq (0,0)$. Show that there exists $\epsilon > 0$, such that $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ is injective.

8. Find ∇f if:

   (a) $f(x,y,z) = \log(z + \sin(y^2 - x))$.

   (b) $f(X) = \|X\|$ where $X = (x,y,z)$.

   (c) $f(X) = \dfrac{1}{\|X\|}$.

[4] https://pennance.us

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show, using the definition of the derivative, that at any point $p \in \mathbb{R}^3$, $Df(p) = f$.

10. Find the derivative $Df(p)(\vec{h})$ if

    (a) $f(x) = \sin x$.

    (b) $f(x, y) = 2x + \sin y$.

    (c) $f(x, y) = (2x + 3y, 4x + 5y)$.

11. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x, y, z) = \int_{h(x,y,z)}^{k(x,y,z)} g(t)dt.$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $k, l : \mathbb{R}^3 \to \mathbb{R}$ are differentiable. Use the chain rule to show that

$$Df(p)(\vec{h}) = [g \circ k(p)]\nabla k(p) \cdot \vec{h} - [g \circ l(p)]\nabla l(p) \cdot \vec{h}.$$

where $p = (x, y, z)$, and $\vec{h} = (h_1, h_2, h_3)$.

12. Let π be the plane tangent tangent to the surface $x^2+y^2+2z^2 = 4$ at the point $(1, 1, 1)$.

13. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto \|x\|$. Find the directional derivatives $D_{e_i} f(x)$, $1 \leq i \leq 3$.

14. Let $f(x, y) = x^2 + y^2$. Find

    (a) The linear map $Df(1, 1)$.

    (b) The rate of increase of f in the direction of $(3, 4)$ at the point $(1, 1)$.

    (c) An equation of the tangent plane to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

    (d) A normal vector to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

    (e) A normal vector to the level curve of f at $(2, 5)$.

    (f) A tangent vector to the level curve of f at $(2, 5)$.

15. Find equations of the two planes with normal vector $(1, 1, 1)$ and tangent to the sphere $x^2 + y^2 + z^2 = 25$.

16. Find an equation of the plane tangent to the surface $\sin xy+\sin yz+\sin xz$ at the point $(1, \pi/2, 0)$.

17. The temperature at point (x, y) is given by

$$f(x, y) = 10 + 6 \cos x \cos y + 3 \cos 2x + 4 \cos 3y$$

Find the direction of greatest decrease of temperature at the point $\pi/3, \pi/3)$.

18. (a) Evaluate $\dfrac{\partial f}{\partial y}(2, 3)$ if $f(x, y) = \dfrac{x + y}{1 - xy}$.

    (b) Evaluate $\dfrac{\partial f}{\partial x}(0, 0)$ if $f(x, y) = \sqrt{xy}$.

(c) Let $f(x,t) : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,t) = \sin(x - ct)$. Show that f satisfies the wave equation: $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$.

(d) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(x,y) = (y\cos x + y^2, \sin x + 2xy - 2y)$. Find a function Φ such that $F = \nabla\Phi$.

(e) Find the line integral of F along the parametrized curve path
$\alpha(t) = (1 + \cos t, 2 + \sin t), \quad 0 \le t \le 2\pi$.

19. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by $h(x,t) = e^{-t}\cos(x/c)$. Show that h satisfies the heat equation: $\dfrac{\partial h}{\partial t} = c^2 \dfrac{\partial^2 h}{\partial x^2}$.