

Scalar Functions of Several Variables

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1. Definition.

Let $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. Let $\vec{u}$ a unit vector in $\mathbb{R}^n$. The *directional derivative* of f at a point $p \in \Omega$ in the *direction* u is defined by

$$D_{\vec{u}}f(p) = \lim_{t \rightarrow 0} \frac{f(p + t\vec{u}) - f(p)}{t}$$

2. Interpretation. $D_{\vec{u}}f(p)$ is just the rate of change of f in the direction u .

Remark. If, in the definition, we allow $\|u\| = c$, then $D_{\vec{u}}f(p)$ is the rate of change of f as seen by a point at p moving with speed c in the direction u .

3. Let $f(x, y) = x^2 + y$. Let P be the point $(1, 3)$ and $\vec{u} = (1, 0)$.

Compute $D_{\vec{u}}f(P)$ from the definition.

Solution.

$$\begin{aligned} D_{\vec{u}}f(P) &= \lim_{t \rightarrow 0} \frac{f(P + t\vec{u}) - f(P)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(1 + t, 3) - f(1, 3)}{t} \\ &= \lim_{t \rightarrow 0} \frac{1 + 2t + t^2 + 3 - 4}{t} \\ &= \lim_{t \rightarrow 0} \frac{2t + t^2}{t} \\ &= 2. \end{aligned}$$

4. Definition.

Let $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$ and $\vec{e}_3 = (0, 0, 1)$ be the standard basis vectors for $\mathbb{R}^3$. The corresponding directional derivatives:

$$D_{\vec{e}_1}f(p) = \lim_{t \rightarrow 0} \frac{f(p_1 + t, p_2, p_3) - f(p_1, p_2, p_3)}{t}$$

$$D_{\vec{e}_2}f(p) = \lim_{t \rightarrow 0} \frac{f(p_1, p_2 + t, p_3) - f(p_1, p_2, p_3)}{t}$$

$$D_{\vec{e}_3}f(p) = \lim_{t \rightarrow 0} \frac{f(p_1, p_2, p_3 + t) - f(p_1, p_2, p_3)}{t}$$

are called the *partial derivatives*.

5. Alternative notation:

$$D_{\vec{e}_1}f(p) = D_1f(p) = \frac{\partial f}{\partial x}(p) = f_x(p)$$

$$D_{\vec{e}_2}f(p) = D_2f(p) = \frac{\partial f}{\partial y}(p) = f_y(p)$$

$$D_{\vec{e}_3}f(p) = D_3f(p) = \frac{\partial f}{\partial z}(p) = f_z(p)$$

6. Remark.

Notice that partial derivatives can be evaluated as ordinary derivatives of a function of a single variable. For example, the student should verify that

$$D_1f(p_1, p_2, p_3) = [t \mapsto f(t, p_2, p_3)]'(p_1).$$

7. Example.

Evaluate $\frac{\partial f}{\partial y}(2, 3)$ if

$$f(x, y) = \frac{|x - 2|}{1 + \sin(y^2 + \tan xy)}.$$

Solution.

$$\begin{aligned} D_2f(2, 3) &= [y \mapsto f(2, y)]'(3) \\ &= [y \mapsto 0]'(3) \\ &= 0 \end{aligned}$$

8. Definition.

If the partial derivatives exist, the vector $(D_1f(p), D_2f(p), \dots, D_nf(p))$ is called the *gradient* of f at p and is denoted $\nabla f(p)$. In the case $n = 3$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

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9. Definition.

We say that $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is *differentiable* at $p \in \Omega$ if there exists a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}$ and a function ϕ defined for h small enough so that $p + \vec{h} \in \Omega$, so that

$$f(p + \vec{h}) = f(p) + L(\vec{h}) + \phi(\vec{h})$$

and

$$\lim_{\vec{h} \rightarrow 0} \frac{\phi(\vec{h})}{\|\vec{h}\|} = 0 \quad (1)$$

If f is differentiable at p , the linear map L is written $Df(p)$ and so

$$f(p + \vec{h}) = f(p) + Df(p)(\vec{h}) + \phi(\vec{h})$$

10. Lemma.

If $\phi(\vec{h})$ satisfies (1) and k is any unit vector, then

$$\lim_{t \rightarrow 0} \frac{\phi(t\vec{k})}{t} = 0.$$

Proof.

Let $\epsilon > 0$. It follows from (1) that there exists $\delta > 0$ such that $\frac{\phi(\vec{h})}{\|\vec{h}\|} < \epsilon$ whenever $0 < \|\vec{h}\| < \delta$. In particular, if $h = tk$ where k is a unit vector,

$$\begin{aligned} 0 < t < \delta &\implies 0 < \|t\vec{k}\| < \delta \\ &\implies \frac{\phi(t\vec{k})}{t} < \epsilon \end{aligned}$$

and the result follows.

11. Claim.

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, then the derivative of f is unique.

Proof.

If

$$\begin{aligned} f(x + \vec{h}) &= f(x) + L(\vec{h}) + \phi(\vec{h}) \\ &= f(x) + \hat{L}(\vec{h}) + \hat{\phi}(\vec{h}), \end{aligned}$$

where L and $\hat{L}$ are linear and ϕ and $\hat{\phi}$ satisfy (1), then, for any h ,

$$L(\vec{h}) - \hat{L}(\vec{h}) = \hat{\phi}(\vec{h}) - \phi(\vec{h}).$$

Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$, then, for $i = 1, 2$

$$L(t\vec{e}_i) - \hat{L}(t\vec{e}_i) = \hat{\phi}(t\vec{e}_i) - \phi(t\vec{e}_i),$$

and by linearity

$$L(\vec{e}_i) - \hat{L}(\vec{e}_i) = \frac{\hat{\phi}(t\vec{e}_i) - \phi(t\vec{e}_i)}{t}.$$

It now follows from (1) that

$$L(\vec{e}_i) = \hat{L}(\vec{e}_i), \quad 1 \leq i \leq n.$$

Since the linear maps L and $\hat{L}$ are equal on a basis they must be equal.

12. Examples.

(a) If f is constant,

$$Df(p)(\vec{h}) = 0$$

for all $\vec{h} \in \mathbb{R}^2$.

(b) If f is linear, then

$$Df(p) = f$$

Proof.

If f is constant, then $f(p + \vec{h}) = f(p)$ and we may take both L and ϕ to be the zero function in the definition of the derivative. By uniqueness, this is the only choice.

If f is linear,

$$f(a + \vec{h}) = f(a) + L(\vec{h}) + \phi(\vec{h})$$

reduces to

$$f(\vec{h}) = L(\vec{h}) + \phi(\vec{h}).$$

The result follows by taking $L = f$ and $\phi = 0$. Again, this is the only choice.

13. Example.

Let $f(x, y) = x^2 + y^2$. Let $p = (x, y)$ and $\vec{h} = (h, k)$.

$$\begin{aligned} f(p + \vec{h}) &= f(x + h, y + k) \\ &= (x + h)^2 + (y + k)^2 \\ &= f(p) + 2xh + 2yk + h^2 + k^2 \end{aligned}$$

Let

$$\phi(\vec{h}) = h^2 + k^2$$

Since,

$$\lim_{h \rightarrow 0} \frac{\phi(\vec{h})}{\|\vec{h}\|} = 0$$

it follows, by uniqueness, that the linear term

$$L(\vec{h}) = 2xh + 2yk$$

is the derivative at (x, y) . That is,

$$Df(p)(\vec{h}) = 2xh + 2yk.$$

14. Claim.

Let $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}$. If f is differentiable at p then the partial derivatives $D_i f$ exists at p and the derivative is given by

$$\begin{aligned} Df(p)(\vec{h}) &= \sum_{i=1}^n D_i f(p) h_i \\ &= \nabla f(p) \cdot \vec{h}. \end{aligned}$$

Proof (n=2).

Since f is differentiable at p there exist constants a and b such that for h sufficiently small

$$f(p + \vec{h}) = f(p) + ah_1 + bh_2 + \phi(h)$$

Taking $\vec{h} = t\vec{e}_1$ and rearranging yields:

$$\frac{f(p + t\vec{e}_1) - f(p)}{t} = a + \frac{\phi(t\vec{e}_1)}{t}$$

Taking the limit as $t \rightarrow 0$ yields $a = D_1 f(p)$. Similarly $b = D_2 f(p)$ and the result is proven.

15. Warning.

The converse of this is false. Existence of $D_1 f(p)$ and $D_2 f(p)$ at a point $p \in \mathbb{R}^2$ does not guarantee differentiability.

16. Claim.

Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $p \in \Omega$. If $D_1 f$ and $D_2 f$ are continuous at p , then f is differentiable at p .

Proof.

See S. Lang, *Calculus of Several Variables*.

17. Example.

Find the derivative $Df(p)(\vec{h})$ if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = 2x + \sin y$.

Solution.

Let $p = (x, y)$ and $\vec{h} = (h_1, h_2)$

$$\begin{aligned} Df(p)(h_1, h_2) &= \nabla f(x, y) \cdot \vec{h} \\ &= (2, \cos y) \cdot (h_1, h_2) \\ &= 2h_1 + (\cos y)h_2 \end{aligned}$$

18. Remark.

Substituting $X = p + \vec{h}$ in

$$f(p + \vec{h}) = f(p) + Df(p)(\vec{h}) + \phi(\vec{h})$$

it is seen for $\vec{h}$ small that

$$\begin{aligned} f(X) &\approx f(p) + Df(p)(X - p) \\ &= f(p) + \nabla f(p) \cdot (X - p) \end{aligned}$$

If $X = (x, y)$ and $p = (a, b)$ then

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(p)(y - b)$$

We (hopefully) see that near $p = (a, b)$ the graph of f is approximated by the plane

$$z = f(p) + D_1 f(p)(x - a) + D_2 f(p)(y - b) \quad (2)$$

19. Definition.

If $z = f(x, y)$ is differentiable at a point $p \in \mathbb{R}^2$, then the subset of $\mathbb{R}^3$ determined by (2) is called the *tangent plane* to the graph of f at a point p .

I.e., the tangent plane at p is translation translation of the graph of the derivative $Df(p)$ by the vector $(p, f(p))$.

20. Example.

Find the tangent plane to the graph of $f(x, y) = 3x + 2y$ at the point $p = (1, 1)$.

Solution.

Since $\nabla f(p) = (3, 2)$ and $f(p) = 5$,

$$\begin{aligned} z - f(p) &= Df(p)(X - p) \\ \iff z - 5 &= \nabla f(p) \cdot (x - 1, y - 1) \\ \iff z - 5 &= 3x - 3 + 2y - 2 \\ \iff z &= 3x + 2y \end{aligned}$$

Naturally, the tangent plane to a plane at any point should be the same plane!

21. Claim.

Let f be differentiable at $X \in \mathbb{R}^2$. Then the directional derivative $D_u(f)$ is linear in u , moreover

$$D_u f(X) = Df(X)(u).$$

Proof.

Let $\vec{u}$ be a non zero vector in $\mathbb{R}^2$. Since f is differentiable at X , there exists a function ϕ such that

$$f(X + t\vec{u}) = f(X) + Df(X)(t\vec{u}) + \phi(t\vec{u})$$

and

$$\lim_{t \rightarrow 0} \frac{\phi(t\vec{u})}{t} = 0.$$

By linearity of the derivative,

$$\frac{\phi(t\vec{u})}{t} = \frac{f(X + t\vec{u}) - f(X)}{t} - Df(X)(\vec{u})$$

and the result follows by taking the limit as $t \rightarrow 0$.

22. Corollary.

$$D_{\vec{u}}f(p) = \nabla f(p) \cdot \vec{u}.$$

23. Corollary.

$\nabla f(p)$ points in the direction of maximum rate of change of f .

Proof.

By the Cauchy Schwartz inequality

$$|D_{\vec{u}}f(p)| \leq \|\nabla f(p)\| \|\vec{u}\|.$$

If $\vec{u}$ is a unit vector, this implies

$$-\|\nabla f(p)\| \leq D_{\vec{u}}f(p) \leq \|\nabla f(p)\|.$$

Since $\nabla f(P) \cdot \vec{u} = \|\nabla f(P)\| \|\vec{u}\| \cos \theta$ the maximum is achieved when $\vec{u}$ and $\nabla f(p)$ point in the same direction. Moreover,

$$\max\{D_{\vec{u}}f(p) : \|\vec{u}\| = 1\} = \|\nabla f(p)\|.$$

24. Example.

Let $f(x, y) = x^2 + y^2$. Find

- The linear map $Df(1, 1)$.
- The rate of increase of f in the direction of $(3, 4)$ at the point $(1, 1)$.
- An equation of the tangent plane to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.
- A normal vector to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

Solution.

(a)

$$\begin{aligned} Df(1, 1)(h_1, h_2) &= \nabla f(1, 1) \cdot (h_1, h_2) \\ &= 2h_1 + 2h_2. \end{aligned}$$

- (b) If u is a unit vector, the rate of increase of f in the direction $[u]$ at a point p is the directional derivative $D_u f(p) = Df(p)(u)$. Taking $u = (\frac{3}{5}, \frac{4}{5})$,

$$\begin{aligned} Df(1,1)(\frac{3}{5}, \frac{4}{5}) &= 2 \cdot \frac{3}{5} + 2 \cdot \frac{4}{5} \\ &= \frac{14}{5}. \end{aligned}$$

- (c) The tangent plane to $z = f(x, y)$

when $p = (1, 1)$ is given by

$$\begin{aligned} z &= f(1, 1) + Df(1, 1)((x, y) - (1, 1)) \\ &= 2 + 2(x - 1) + 2(y - 1) \end{aligned}$$

Hence the tangent plane at $(1, 1)$ is:

$$2x + 2y + (-1)z + 2 = 0.$$

- (d) A normal n to the surface at $(1, 1, 2)$ is just a normal to the tangent plane at that point. Thus $n = (2, 2, -1)$ is a normal vector.