Formulae for curvature

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1. Definition.

   If $\hat{\gamma}(s)$ is a unit speed curve, then curvature k is defined by

   $$k(s) = \|\gamma''(s)\|.$$

2. Claim.

   Let $\gamma(t)$ be a regular curve with speed $v(t) = \|\gamma'(t)\|$.

   $$k(t) = \frac{1}{v}\left\|\frac{d}{dt}\frac{\gamma'(t)}{v}\right\|.$$

   Proof.

   Let
   $$s(t) = \int_{t_0}^{t} \|\gamma'(t)\|$$
   and let
   $$v = s'(t) = \|\gamma'(t)\|.$$

   Then
   $$\hat{\gamma} = \gamma \circ s^{-1}$$
   is a unit speed reparametrization of γ. Hence

   $$\begin{aligned}
   &\hat{\gamma}(s(t)) = \gamma(t) \\
   \implies\ &\hat{\gamma}'(s(t))s'(t) = \gamma'(t) \\
   \implies\ &\hat{\gamma}'(s(t)) = \frac{\gamma'(t)}{s'(t)} \\
   \implies\ &\hat{\gamma}''(s(t))s'(t) = \frac{d}{dt}\frac{\gamma'(t)}{s'(t)} \\
   \implies\ &\hat{\gamma}''(s(t)) = \frac{1}{v}\frac{d}{dt}\frac{\gamma'(t)}{v} \\
   \implies\ &k = \frac{1}{v}\left\|\frac{d}{dt}\frac{\gamma'(t)}{v}\right\|.
   \end{aligned}$$

3. Example.

   Let
   $$\gamma(t) = (\cos t, \sin t, t).$$

   Then,
   $$\gamma'(t) = (-\sin t, \cos t, 1).$$

   Then
   $$\begin{aligned}
   v &= \|\gamma'(t)\| \\
   &= \|(-\sin t, \cos t, 1)\| \\
   &= \sqrt{2}.
   \end{aligned}$$

   Hence
   $$\begin{aligned}
   \kappa &= \frac{1}{v(t)}\left\|\frac{d}{dt}\left(\frac{\gamma'(t)}{v(t)}\right)\right\| \\
   &= \tfrac{1}{2}\|\gamma''(t)\| \\
   &= \tfrac{1}{2}.
   \end{aligned}$$

4. Lemma (Triple vector product).

   Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, then
   $$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

5. Claim.

   Let $\gamma(t)$ be a regular curve.
   $$k = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}.$$

   Proof.
   $$\begin{aligned}
   k &= \frac{1}{v}\left\|\frac{d}{dt}\frac{\gamma'(t)}{v}\right\| \\
   &= \frac{1}{v}\left\|\frac{\gamma''}{v} + \gamma'\frac{d}{dt}\left(\frac{1}{v}\right)\right\| \\
   &= \frac{1}{v}\left\|\frac{\gamma''}{v} - \gamma'\frac{\gamma'\cdot\gamma''}{\|\gamma'\|^3}\right) \\
   &= \frac{1}{v^4}\left\|\|\gamma'\|^2\gamma'' - (\gamma'\cdot\gamma'')\gamma'\right\| \\
   &= \frac{1}{v^4}\left\|(\gamma'\cdot\gamma')^2\gamma'' - (\gamma'\cdot\gamma'')\gamma'\right\| \\
   &= \frac{1}{v^4}\left\|\gamma' \times (\gamma'' \times \gamma')\right\| \\
   &= \frac{1}{v^4}\|\gamma'\|\|\gamma'' \times \gamma'\| \\
   &= \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}.
   \end{aligned}$$

[1] https://pennance.us

Remark: This result can also be proven using the Frenet formulae.

6. Corollary.

   Let $\gamma(t)$ be a regular curve in $\mathbb{R}^2$. Then,

$$k = \frac{|\gamma' \wedge \gamma''|}{\|\gamma'\|^3}. \tag{1}$$

   where $\gamma' \wedge \gamma''$ is the oriented area of the parallelogram determined by γ', γ''.

   Proof.

   Recall that if $b, c \in \mathbb{R}^2$, then

$$b \wedge c = b_1 c_2 - b_2 c_1.$$

   Let $a = (a_1, a_2) \in \mathbb{R}^2$. and $a^\perp = (a_2, -a_1)$ then simple computation yields a two dimensional analogue of the triple product formula.

$$(a \cdot c)b - (a \cdot b)c = (b \wedge c)\ a^\perp.$$

   (1) follows by replacing the triple vector product in the previous proof with its two dimensional analogue.

   Alternatively, it suffices to note observe that if $\gamma = (\gamma_1, \gamma_2, 0)$ then

$$\gamma' \times \gamma'' = (0, 0, \gamma' \wedge \gamma'').$$

7. Versions of (1)

   (a) If $\gamma(t) = (x(t), y(t))$ then (**??**) can be written

$$k = \frac{|x'y'' - x''y'|}{\left[\sqrt{x'^2 + y'^2}\right]^3}$$

   (b) In the case of the graph of a function, $y = f(x)$, parametrization by

$$\gamma(t) = (t, f(t))$$

   yields

$$k = \frac{|f''|}{\left[\sqrt{1 + f'^2}\right]^3}.$$

   (c) In the case of a curve given in polar coordinates $r = r(\theta)$, parametrization by

$$\gamma(t) = (r(t)\cos t, r(t)\sin t)$$

   yields

$$k = \frac{|r^2 + 2r'^2 - rr''|}{\left[\sqrt{r^2 + r'^2}\right]^3}$$