Change of Variables Theorem

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1. One dimensional case.

   Let $I \subseteq \mathbb{R}$ be an interval and

   $$x : [a, b] \to I$$

   be a differentiable function with continuous derivative. Suppose that $f : I \to \mathbb{R}$ is continuous. Then

   $$\int_{x(a)}^{x(b)} f(x)\, dx = \int_a^b f \circ x(\theta) \cdot x'(\theta)\, d\theta.$$

   Proof.

   Let F be an antiderivative of f. Then $F \circ x$ is an antiderivative of $f \circ x \cdot x'$. It follows that both sides are equal to $(F \circ x)(b) - (F \circ x)(a)$.

2. Example.

   Let $x : [0, \frac{\pi}{2}] \to [0, 1]$ be given by

   $$x(\theta) = \sin \theta.$$

   Then,

   $$\begin{aligned} \int_0^1 \sqrt{1 - x^2}\, dx &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \theta} \cdot \cos \theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= \frac{\pi}{4}. \end{aligned}$$

3. Two dimensional case.

   Let $R \subseteq \mathbb{R}^2$ be a rectangle and

   $$X : R \to \mathbb{R}^2$$

   a function which is C^1-invertible on the interior of R. Let $f : X(R) \to \mathbb{R}$ be continuous, then

   $$\begin{aligned} \int_{X(R)} f &= \int_R f \circ X \cdot |\det\ DX| \\ &= \iint_R f \circ X(u, v) |\det DX(u, v)|\, du\, dv. \end{aligned}$$

4. Example.

   In the (r, θ) plane let R be the rectangle $[0, \rho] \times [0, \frac{\pi}{2}]$

   Let $X : R \to \mathbb{R}^2$ be given by

   $$X(r, \theta) = (r \cos \theta, r \sin \theta)$$

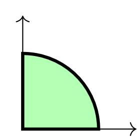
   Then DX has matrix

   $$J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

   and

   $$\det DX = \det J = r.$$

   Then, X maps the rectangle R bijectively onto the sector $X(R)$ shown below.

   If $f : X(R) \to \mathbb{R}$ is a continuous function on the sector $X(R)$ then

   $$\iint_{X(R)} f(x, y)\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{\rho} f(r \cos \theta, r \sin \theta) r\, dr\, d\theta.$$

[1] https://pennance.us