

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

EXERCISES ON VECTOR GEOMETRY
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1. If a diagonal and an edge of a cube intersect. Find the angle between them. 0By proportionality, the angle is independent of the length of the sides. Hence, it can be assumed that the cube is the standard unit cube $[0, 1]^3$. The angle θ between the diagonal vector $\alpha = (1, 1, 1)$ and the vector $\beta = (1, 0, 0)$ is given by

$$\begin{aligned}\theta &= \cos^{-1} \frac{\alpha \cdot \beta}{|\alpha||\beta|} \\ &= \cos^{-1} \frac{1}{\sqrt{3}}.\end{aligned}$$

2. Let

$$\begin{aligned}\vec{\alpha} &= (-1, 2, -3) \\ \vec{\beta} &= (3, 4, 5).\end{aligned}$$

- (a) Evaluate $\langle \vec{\alpha}, \vec{\beta} \rangle$.

0

$$\langle \vec{\alpha}, \vec{\beta} \rangle = (-1)(3) + (2)(4) + (-3)(5) = -10.$$

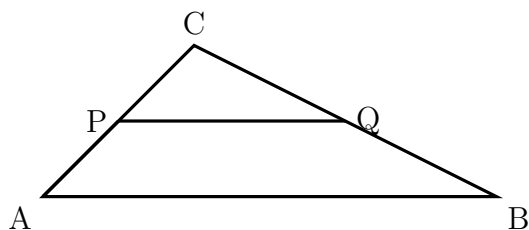
- (b) Find the projection of $\vec{\alpha}$ on $\vec{\beta}$.

$$\begin{aligned}P_{\vec{\beta}}(\vec{\alpha}) &= \frac{\langle \vec{\alpha}, \vec{\beta} \rangle}{\langle \vec{\beta}, \vec{\beta} \rangle} \vec{\beta} \\ &= \frac{-10}{50} (3, 4, 5) \\ &= -\frac{1}{5} (3, 4, 5).\end{aligned}$$

- (c) Evaluate $\vec{\alpha} \times \vec{\beta}$.

$$\vec{\alpha} \times \vec{\beta} = (22, -4, -10).$$

3. Use vectors to show that if P and Q are the midpoints of CA and CB respectively, then PQ is parallel to AB



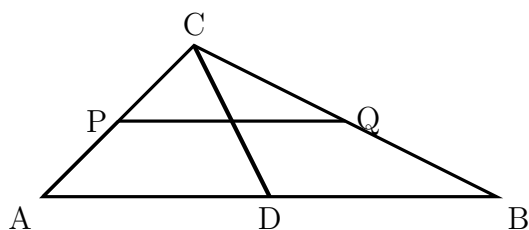
Solution.

Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B, C respectively. Then,

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \left(\frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right) - \left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{c}\right) \\ &= \frac{1}{2}(\vec{b} - \vec{a}) \\ &= \frac{1}{2}\overrightarrow{AB}.\end{aligned}$$

Hence, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$.

4. Use vectors to show that if P and Q are the midpoints of CA and CB respectively, then the line through C which bisects AB also bisects PQ



Solution.

Take the origin at A . Let $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{AC}$ be the position vectors of B and C respectively. The midpoint D of AB has position vector $\frac{\vec{b}}{2}$ and so the line $\overleftrightarrow{CD}$ has equation $\vec{r}(t) = (1-t)\vec{c} + \frac{t}{2}\vec{b}$. The midpoint M of PQ has position vector

$$\begin{aligned}\vec{m} &= \frac{1}{2}\vec{c} + \frac{1}{2}\left(\frac{1}{2}\vec{c} + \frac{1}{2}\vec{b}\right) \\ &= \frac{1}{2}\vec{c} + \frac{1}{4}\vec{b}.\end{aligned}$$

Since $\vec{r}(1/2) = \vec{m}$, the line $\overleftrightarrow{CD}$ bisects PQ .

5. Let a, b be non zero vectors in $\mathbb{R}^3$ and such that $\langle a, b \rangle = 0$. Let α, β and γ are real numbers such that $\alpha a + \beta b + \gamma a \times b = O$. Show $\alpha = \beta = \gamma = 0$.

Taking the scalar product of both sides with $\vec{a}$ and using $\langle a, b \rangle = 0$ and $\langle a, a \times b \rangle = 0$ yields $\beta \langle b, b \rangle = 0$. Since $b \neq \vec{0}$ it must be that $\beta = 0$. Similarly, taking the scalar product with $\vec{a}$ shows that $\alpha = 0$. Hence $\gamma a \times b = O$. Since $\vec{a}$ is orthogonal to $\vec{b}$ and $\vec{a}$ and $\vec{b}$ are non zero, it follows (by the Lagrange identity) that $\gamma = 0$.

6. Let

$$\begin{aligned}\vec{\alpha} &= (3, 1, -1) \\ \vec{\beta} &= (\lambda - 1, 7, -\frac{7}{22}\lambda).\end{aligned}$$

- (a) Find λ if $\vec{\beta}$ and $\vec{\alpha}$ are parallel.

$\vec{\beta}$ and $\vec{\alpha}$ are parallel if and only if there is a constant $c \neq 0$ such that $\vec{\beta} = c\vec{\alpha}$. Comparing the second components of the vectors reveals that necessarily, $c = 7$ and so $\vec{\beta} = (21, 7, -7)$. It follows that $\lambda = 22$.

- (b) Find λ if $\vec{\beta}$ and $\vec{\alpha}$ are orthogonal.

$$\begin{aligned}\langle \alpha, \beta \rangle = 0 &\iff 3(\lambda - 1) + (1)(7) + (-1)(-\frac{7}{22}\lambda) = 0 \\ &\iff \lambda = -\frac{88}{73}.\end{aligned}$$

7. Let $\vec{a}$ be a non zero vector in $\mathbb{R}^3$. If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, show that $\vec{b} = \vec{c}$.

Let $\vec{w} = \vec{b} - \vec{c}$. From the bilinearity of the cross product we have

$$\begin{aligned}\vec{a} \times \vec{b} = \vec{a} \times \vec{c} &\iff \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0} \\ &\iff \vec{a} \times (\vec{b} - \vec{c}) = \vec{0} \\ &\iff \vec{a} \times \vec{w} = \vec{0}.\end{aligned}$$

Similarly, using the bilinearity of the scalar product $\vec{a} \cdot \vec{w} = 0$. By the Lagrange identity,

$$\|\vec{a} \times \vec{w}\|^2 = \|\vec{a}\|^2 \|\vec{w}\|^2 - (\vec{a} \cdot \vec{w})^2.$$

Since $\vec{a} \times \vec{w} = \vec{0}$ and $\vec{a} \cdot \vec{w} = 0$ it follows that $\|\vec{a}\|^2 \|\vec{w}\|^2 = 0$. But $\|a\| \neq 0$. It follows that $\|\vec{w}\| = 0$. This can only happen if $\vec{w} = \vec{0}$. That is, $\vec{b} - \vec{c} = \vec{0}$, and so $\vec{b} = \vec{c}$.

8. The *skew product* of two vectors in $\mathbb{R}^2$ is defined by the formula:

$$(x_1, y_1) \wedge (x_2, y_2) = x_1 y_2 - x_2 y_1.$$

- (a) Show that $(c_1 v_1) \wedge (c_2 v_2) = c_1 c_2 (v_1 \wedge v_2)$ for any scalars c_1, c_2 and any vectors v_1, v_2 .

Solution.

Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Then $c_1 v_1 = (c_1 x_1, c_1 y_1)$ and $c_2 v_2 = (c_2 x_2, c_2 y_2)$. and so

$$\begin{aligned} c_1 v_1 \wedge c_2 v_2 &= (c_1 x_1 \cdot c_2 y_2 - c_2 x_2 \cdot c_1 y_1) \\ &= c_1 c_2 (x_1 y_2 - x_2 y_1) \\ &= c_1 c_2 v_1 \wedge v_2. \end{aligned}$$

- (b) Compute $(\cos \alpha, \sin \alpha) \wedge (\cos \beta, \sin \beta)$.

$$\begin{aligned} (\cos \alpha, \sin \alpha) \wedge (\cos \beta, \sin \beta) &= \cos \alpha \sin \beta - \sin \alpha \cos \beta \\ &= \sin(\beta - \alpha). \end{aligned}$$

- (c) What geometric circumstance determines the sign of the skew product?

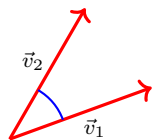
Using the same argument as in (b) The wedge product in polar coordinates takes the form

$$(r \cos \alpha, r \sin \alpha) \wedge (r' \cos \beta, r' \sin \beta) = rr' \sin(\beta - \alpha).$$

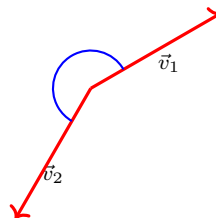
It follows that

$$\begin{aligned} (r \cos \alpha, r \sin \alpha) \wedge (r' \cos \beta, r' \sin \beta) > 0 &\iff \sin(\beta - \alpha) > 0 \\ &\iff 0 < \beta - \alpha < \pi. \end{aligned}$$

i. $0 < \beta - \alpha < \pi$.



ii. $\pi < \beta - \alpha < 2\pi$.



- (d) Let $\vec{a}, \vec{b}$ be non zero vectors in $\mathbb{R}^2$. Show that $\vec{a}$ and $\vec{b}$ are linearly independent if and only if $\vec{a} \wedge \vec{b} \neq 0$.

Solution.

It suffices (by contraposition) to show that $(\vec{a}, \vec{b})$ is an dependent list in $\mathbb{R}^2$ if and

only if $\vec{a} \wedge \vec{b} = 0$.

Suppose $\vec{a}, \vec{b}$ are dependent non zero vectors. Then (exercise) $\vec{a} = c\vec{b}$ for some non zero constant c . Hence,

$$\begin{aligned}\vec{a} \wedge \vec{b} &= (c\vec{b}) \wedge \vec{b} \\ &= c(\vec{b} \wedge \vec{b}) \\ &= 0.\end{aligned}$$

Conversely, suppose $\vec{a} \wedge \vec{b} = 0$, i.e., $a_1b_2 - a_2b_1 = 0$. Since $\vec{b}$ is non zero, either $b_1 \neq 0$ or $b_2 \neq 0$. If $b_2 \neq 0$ then

$$\begin{aligned}(a_1, a_2) &= \left(\frac{a_1b_2}{b_2}, a_2 \right) \\ &= \left(\frac{a_2b_1}{b_2}, a_2 \right) \\ &= a_2 \left(\frac{b_1}{b_2}, 1 \right) \\ &= \frac{a_2}{b_2} (b_1, b_2).\end{aligned}$$

and so $(\vec{a}, \vec{b})$ is a dependent list. The case $b_1 \neq 0$ is similar.

9. Find a vector parallel to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$. Find an equation for the line of intersection of the two planes.

Solution.

The planes have normal vectors $n_1 = (2, -1, 1)$ and $n_2 = (3, 1, 1)$. The cross product $\vec{v} = n_1 \times n_2 = (-2, 1, 5)$ is a direction vector to the line of intersection of the two planes. Putting $x = 0$ in the equations of the planes and solving for the other coordinates yields a point $P = (0, \frac{1}{2}, \frac{3}{2})$. in the intersection of the planes. Hence, an equation for the line of intersection is

$$\begin{aligned}X(t) &= P + tv, \quad t \in \mathbb{R} \\ &= (0, \frac{1}{2}, \frac{3}{2}) + t(-2, 1, 5) \quad t \in \mathbb{R}.\end{aligned}$$

10. (a) Write the line $2x - 4y + 5 = 0$ in the vector form $P + [u]$ where $P \in \mathbb{R}^2$ and $\vec{u}$ is a unit vector.

Solution.

Let $P = (0, \frac{5}{4})$. Since $(2, -4)$ is normal to the line, $\vec{v} = (4, 2)$ is a direction vector. Take $u = \frac{\vec{v}}{\|\vec{v}\|} = (\frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}}) = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$.

- (b) Let $P = (4, 3)$ and $\vec{u} = (1, 2)$, write the image of the curve $\gamma(t) = P + t\vec{u}$, $t \in \mathbb{R}$ in the Cartesian form $y = mx + b$.

Solution.

$$\gamma(t) = (4, 3) + t(1, 2).$$

$$x = 4 + t$$

$$y = 3 + 2t$$

Eliminating t gives $y = 3 + 2(x - 4) = 2x - 5$.

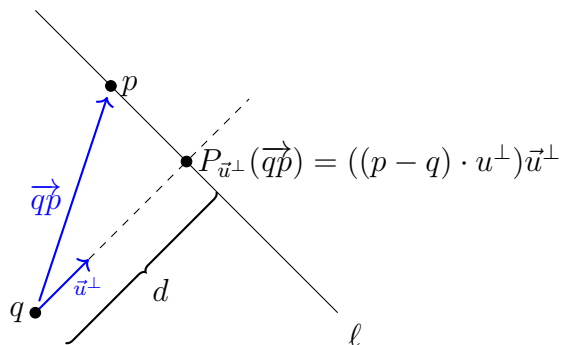
11. (a) Find a unit vector $u^\perp$ perpendicular to the line ℓ given by $ax + by + c = 0$ where a and b are not both zero.

Since $(a, b) \perp \ell$, we may take $u^\perp = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$.

- (b) Let p be the point $(0, -\frac{c}{b}) \in \ell$. and let $q = (\xi, \eta) \in \mathbb{R}^2$. Compute the projection $P_{u^\perp}(\vec{pq})$ of the free vector $\vec{pq}$.

Since $u^\perp$ is a unit vector,

$$\begin{aligned} P_{u^\perp}(\vec{pq}) &= \langle q - p, u^\perp \rangle u^\perp \\ &= \left\langle (\xi, \eta) - \left(0, -\frac{c}{b}\right), u^\perp \right\rangle u^\perp \\ &= \left\langle \left(\xi, \eta + \frac{c}{b}\right), \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right) \right\rangle u^\perp \\ &= \frac{a\xi + b\eta + c}{\sqrt{a^2 + b^2}} u^\perp. \end{aligned}$$



- (c) Let $d = |P_{u^\perp} \vec{pq}|$. What does d represent geometrically?

d is the perpendicular distance of the point $q = (\xi, \eta)$ to the line ℓ .

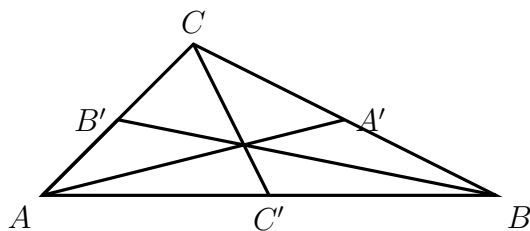
12. Let P be a point on the plane $2x + 3y + 4z = 5$. Find the projection of P on the normal vector to the plane.

Let $n = (2, 3, 4)$ and $\hat{n} = \frac{n}{\|n\|} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right)$ and Let $P = (\xi, \eta, \zeta)$ be a point on the

plane

$$\begin{aligned}\text{Proj}_n(P) &= (P \cdot \hat{n}) \hat{n} \\ &= \left(\frac{2\xi + 3\eta + 4\zeta}{\sqrt{29}} \right) \hat{n} \\ &= \frac{5}{\sqrt{29}} \hat{n}\end{aligned}$$

13. Use vectors to show that the medians of a triangle are concurrent.

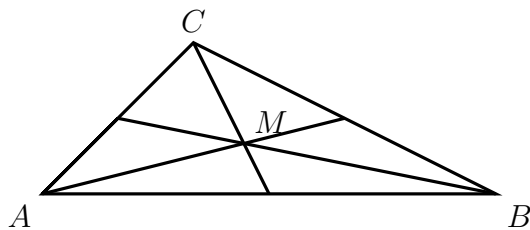


The segment connecting the A to the midpoint A' of BC) has equation

$$\vec{r} = (1-t)\vec{a} + t \frac{\vec{b} + \vec{c}}{2} \quad 0 \leq t \leq 1.$$

Putting $t = \frac{2}{3}$ shows that the *centroid* M with position vector $\vec{m} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$ lies on this median. Similarly, or by symmetry, M lies on the other two medians. Hence the medians intersect at M .

14. Consider the $\triangle ABC$ in which $A = (1, 1, 1)$, $B = (2, 1, 1)$, and $C = (3, 4, 4)$. Find the cosine of the angle between the sides AB and the segment connecting A with the centroid M .



Solution.

The centroid

$$\begin{aligned}M &= \frac{A + B + C}{3} \\ &= (2, 2, 2)\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AB} &= B - A \\
&= (1, 0, 0) \\
\overrightarrow{AM} &= M - A \\
&= (1, 1, 1)
\end{aligned}$$

Hence

$$\begin{aligned}
\cos(\angle MAB) &= \frac{\overrightarrow{AM} \cdot \overrightarrow{AB}}{\|\overrightarrow{AM}\| \|\overrightarrow{AB}\|} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

15. Two vectors $\vec{a}$ and $\vec{b}$ are *dependent* if there exist constants, c_1, c_2 , not both zero, such that $c_1\vec{a} + c_2\vec{b} = \mathbf{0}$. If $\vec{a} \wedge \vec{b} = 0$, show that $\vec{a}$ and $\vec{b}$ are dependent.

Solution.

Since $\vec{a} \wedge \vec{b} = 0$, it follows that $a_1b_2 = a_2b_1$. If $\vec{b} = 0$ then $\vec{a}, \vec{b}$ are dependent since $17\vec{b} + 0\vec{a} = \vec{0}$. Hence we may assume that $\vec{b}$ is non zero. Therefore, either $b_1 \neq 0$ or $b_2 \neq 0$. If $b_2 \neq 0$, then

$$\begin{aligned}
(a_1, a_2) &= \left(\frac{a_2b_1}{b_2}, a_2 \right) \\
&= a_2 \left(\frac{b_1}{b_2}, 1 \right) \\
&= \frac{a_2}{b_2} (b_1, b_2).
\end{aligned}$$

and so $(\vec{a}, \vec{b})$ is a dependent list. The case $b_1 \neq 0$ is similar.

16. Consider the line ℓ given by $ax + by + c = 0$. Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be points on ℓ . Let $\vec{v} = (a, b)$. Show that $\vec{v} \cdot \vec{pq} = 0$.

Solution.

Recall $\vec{pq} = q - p = (x_2 - x_1, y_2 - y_1)$. Since $p, q \in \ell$

$$\begin{aligned}
ax_1 + by_1 + c &= 0 \\
ax_2 + by_2 + c &= 0
\end{aligned}$$

Subtraction yields

$$a(x_2 - x_1) + b(y_2 - y_1) = 0.$$

I.e.,

$$\vec{v} \cdot \vec{pq} = 0$$

17. (a) Consider the plane π given by $ax + by + cz + d = 0$. Let $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ be points in π . Let $\vec{v} = (a, b, c)$. Show that $\vec{v} \cdot \vec{pq} = 0$.

- (b) Find a vector parallel to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$.

Solution.

Recall $\vec{pq} = q - p = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$. Since $p, q \in \ell$

$$ap_1 + bp_2 + cp_3 + d = 0$$

$$aq_1 + bq_2 + cq_3 + d = 0$$

Subtraction yields

$$a(p_1 - q_1) + b(p_2 - q_2) + c(p_3 - q_3) = 0.$$

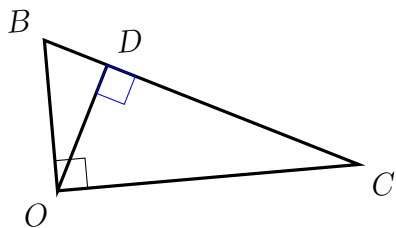
Hence,

$$\vec{v} \cdot \vec{pq} = 0.$$

It follows from part (a) that the planes have normal vectors $n_1 = (2, -1, 1)$ and $n_2 = (3, 1, 1)$ respectively. Hence, the cross product $\vec{v} = n_1 \times n_2 = (-2, 1, 5)$ is a direction vector for the line of intersection of the two planes.

18. Let $O = (0, 0, 0)$, B and C be the vertices of a right triangle in $\mathbb{R}^3$ with hypotenuse BC . Let D be the foot of the perpendicular from O to BC . Let $\vec{b}, \vec{c}, \vec{d}$ be the position vectors of B, C, D respectively. Find (in terms of B and C the constant λ such that

$$\vec{d} = (1 - \lambda)\vec{b} + \lambda\vec{c} \quad (1)$$



Solution.

Taking the dot product of (1) with $\vec{b}$ and with $\vec{c}$ and noting that $\vec{b} \cdot \vec{c} = 0$ yields

$$\vec{b} \cdot \vec{d} = (1 - \lambda)\|\vec{b}\|^2$$

$$\vec{c} \cdot \vec{d} = \lambda\|\vec{c}\|^2$$

Subtracting gives

$$(\vec{b} - \vec{c}) \cdot \vec{d} = (1 - \lambda)\|\vec{b}\|^2 - \lambda\|\vec{c}\|^2$$

Since $\vec{d}$ is perpendicular to $\vec{b} - \vec{c}$ the left hand side is zero. Hence

$$(1 - \lambda)\|\vec{b}\|^2 - \lambda\|\vec{c}\|^2 = 0$$

$$\implies \lambda = \frac{\|\vec{b}\|^2}{\|\vec{b}\|^2 + \|\vec{c}\|^2}$$

$$\implies 1 - \lambda = \frac{\|\vec{c}\|^2}{\|\vec{b}\|^2 + \|\vec{c}\|^2}$$

and so

$$\vec{d} = \frac{\|\vec{b}\|^2}{\|\vec{c}\|^2 + \|\vec{b}\|^2} \vec{b} + \frac{\|\vec{c}\|^2}{\|\vec{c}\|^2 + \|\vec{b}\|^2} \vec{c}.$$

19. Express $(3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta})$ as a scalar multiple of $\vec{\alpha} \times \vec{\beta}$.

Solution.

Since the cross product is bilinear and antisymmetric,

$$\begin{aligned}(3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta}) &= 3(\vec{\alpha} \times \vec{\alpha}) + 12(\vec{\alpha} \times \vec{\beta}) + (2\vec{\beta} \times \vec{\alpha}) + 8(2\vec{\beta} \times \vec{\beta}) \\&= 12(\vec{\alpha} \times \vec{\beta}) + 8(\vec{\beta} \times \vec{\alpha}) \\&= 12(\vec{\alpha} \times \vec{\beta}) - 8(\vec{\alpha} \times \vec{\beta}) \\&= 10(\vec{\alpha} \times \vec{\beta}).\end{aligned}$$

20. Show that the midpoints of the sides of a quadrilateral $ABCD$ in $\mathbb{R}^3$ form a parallelogram. **Solution.**

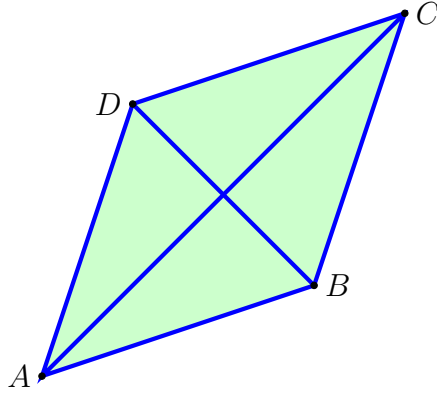
Let

$$\begin{aligned}M_1 &= \frac{A + B}{2} \\M_2 &= \frac{B + C}{2} \\M_3 &= \frac{C + D}{2} \\M_4 &= \frac{D + A}{2}\end{aligned}$$

Then,

$$\begin{aligned}M_1 - M_2 &= \frac{A - C}{2} \\&= M_4 - M_3 \\M_1 - M_4 &= \frac{B - D}{2} \\&= M_2 - M_3\end{aligned}$$

21. Show that the diagonals of a parallelogram intersect at their midpoints.



Let M_1 be the midpoint of AC and M_2 the midpoint of DB . Since $\vec{AB} = \vec{DC}$

$$\begin{aligned}\vec{AM_2} &= \vec{AB} + \frac{1}{2}\vec{BD} \\ &= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{DC} + \frac{1}{2}\vec{BD} \\ &= \frac{1}{2}\vec{AC} \\ &= \vec{AM_1}\end{aligned}$$

Hence, $M_1 = M_2$.

22. Use vectors to show that the diagonals of a parallelogram coincide at the midpoints. Let $ABCD$ be a parallelogram with $\vec{AB} = \vec{DC}$. Let a, b, c, d be the position vectors of the points A, B, C, D respectively. Then

$$\begin{aligned}\vec{AB} &= \vec{DC} \\ \iff \vec{OA} - \vec{OB} &= \vec{OD} - \vec{OC} \\ \iff a - b &= d - c \\ \iff \frac{a + c}{2} &= \frac{b + d}{2}\end{aligned}$$

It follows that the midpoints of diagonals AC and BD coincide.

23. Let $\vec{\alpha} = (2, 6, 3)$ and $\vec{\beta} = (\lambda - 1, 2\lambda + 2, \lambda + 1)$. Find λ if:

- (a) $\vec{\beta}$ and $\vec{\alpha}$ are parallel.

Solution. If $\vec{\beta}$ and $\vec{\alpha}$ are parallel, there exists a constant c such that $\beta = c\alpha$. Hence

$$\begin{aligned}\lambda - 1 &= 2c \\ 2\lambda + 2 &= 6c \\ \lambda + 1 &= 3c.\end{aligned}$$

It follows that $c = 2$ and $\lambda = 5$.

Alternative idea: $\vec{\beta}$ and $\vec{\alpha}$ are parallel if their cross product is zero.

(b) $\vec{\beta}$ and $\vec{\alpha}$ are orthogonal.

Solution.

$$\begin{aligned}\langle \vec{\alpha}, \vec{\beta} \rangle &= 0 \\ \iff 2(\lambda - 1) + 6(2\lambda + 2) + 3(\lambda + 1) &= 0 \\ \iff \lambda &= -\frac{13}{17}.\end{aligned}$$

24. Let $\vec{\alpha} = (-1, 2, -3)$ and $\vec{\beta} = (2, 3, 6)$.

(a) Evaluate $\langle \vec{\alpha}, \vec{\beta} \rangle$.

Solution.

$$\langle \vec{\alpha}, \vec{\beta} \rangle = (-1)(2) + (2)(3) + (-3)(6) = -14.$$

(b) Evaluate $\|\vec{\beta}\|$.

Solution.

$$\|\vec{\beta}\| = \sqrt{2^2 + 3^2 + 6^2} = 7.$$

(c) Find the projection of $\vec{\alpha}$ on $\vec{\beta}$.

$$\begin{aligned}P_{\vec{\beta}}(\vec{\alpha}) &= \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} \vec{\beta} \\ &= \frac{-14}{49} (2, 3, 6) \\ &= \left(\frac{-4}{7}, \frac{-6}{7}, \frac{-12}{7} \right)\end{aligned}$$

(d) Evaluate $\vec{\alpha} \times \vec{\beta}$.

Solution.

$$\begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix} = 21i + 0j - 7k = (21, 0, -7).$$

25. Express $(3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta})$ as a scalar multiple of $\vec{\alpha} \times \vec{\beta}$.

Solution.

Since the cross product is bilinear and antisymmetric,

$$\begin{aligned}(3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta}) &= 3(\vec{\alpha} \times \vec{\alpha}) + 12(\vec{\alpha} \times \vec{\beta}) + (2\vec{\beta} \times \vec{\alpha}) + 8(2\vec{\beta} \times \vec{\beta}) \\ &= 12(\vec{\alpha} \times \vec{\beta}) + 8(\vec{\beta} \times \vec{\alpha}) \\ &= 12(\vec{\alpha} \times \vec{\beta}) - 8(\vec{\alpha} \times \vec{\beta}) \\ &= 10(\vec{\alpha} \times \vec{\beta}).\end{aligned}$$

$$\begin{aligned}
(\alpha + 2\beta) \times (3\alpha + 4\beta) &= (\alpha \times 3\alpha) + (\alpha \times 4\beta) + (2\beta \times 3\alpha) + (2\beta \times 4\beta) \\
&= 3(\alpha \times \alpha) + 4(\alpha \times \beta) + 6(\beta \times \alpha) + 8(\beta \times \beta) \\
&= 4(\alpha \times \beta) + 6(\beta \times \alpha) \\
&= 4(\alpha \times \beta) - 6(\alpha \times \beta) \\
&= -2(\alpha \times \beta).
\end{aligned}$$

26. Determine whether the vectors $(1, -1, -2)$, $(2, 1, -1)$ and $(4, 5, 1)$, are linearly independent.

27. Consider the $\triangle ABC$ in which $A = (1, 2, 3)$, $B = (-1, -2, 2)$, and $C = (2, 3, -2)$.

Find:

- (a) the cosine of the angle between the sides AB and AC
- (b) the length of the side AB .
- (c) the midpoint of side AB .

Solution.

$$\begin{aligned}
\overrightarrow{AB} &= (-2, -4, -1) \\
\|\overrightarrow{AB}\| &= \sqrt{21} \\
\overrightarrow{AC} &= (1, 1, -5) \\
\|\overrightarrow{AC}\| &= \sqrt{27} = 3\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
\cos \angle BAC &= \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|} \\
&= \frac{-1}{3\sqrt{63}}
\end{aligned}$$

28. Let P be a point and n a unit vector in $\mathbb{R}^3$. Let M, N be two distinct points in the plane π through P with normal n . Show that line through M and N is contained in the plane π .

Solution.

The plane π has equation $n \cdot (X - P) = 0$. Since $M, N \in \pi$

$$n \cdot (M - P) = 0$$

$$n \cdot (N - P) = 0$$

Let X be a point on the line containing M and N . Then $X = tM + (1 - t)N$ for some $t \in \mathbb{R}$. Then

$$\begin{aligned} n \cdot (X - P) &= n \cdot (tM + (1 - t)N - n \cdot P) \\ &= tn \cdot M + (1 - t)n \cdot N - n \cdot P \\ &= tn \cdot P + (1 - t)n \cdot P - n \cdot P \\ &= 0 \end{aligned}$$

Hence $X \in \pi$ and so the line is contained in π .

29. Find the perpendicular distance from the point $q = (1, 2, 3)$ to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$.

Solution.

The planes have normal vectors $n = (2, -1, 1)$ and $n' = (3, 1, 1)$. A direction vector for the line L of intersection is $\vec{v} = n \times n' = (-2, 1, 5)$. It is trivial to verify that the point $p = (0, \frac{1}{2}, \frac{3}{2})$ belongs to the intersection. Hence, an equation for the line of intersection is $X(t) = (0, \frac{1}{2}, \frac{3}{2}) + t(-2, 1, 5)$. It is not difficult to show (and is obvious from a picture!!) that the foot of the perpendicular from the point q to the line, is the point

$$F = p + P_{\vec{v}}(q - p).$$

where $P_{\vec{v}}(q - p)$ is the projection of $q - p$ on the the direction vector $\vec{v}$. The distance of q to the line L is given by

$$\begin{aligned} d(q, F) &= d(q - p, F - p) \\ &= d(q - p, P_{\vec{v}}(q - p)). \end{aligned}$$

$$q - p = (1, 2, 3) - (0, \frac{1}{2}, \frac{3}{2}) = (1, \frac{3}{2}, \frac{3}{2})$$

$$\begin{aligned} P_{\vec{v}}(q - p) &= \frac{\langle q - p, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \\ &= \left(\frac{-7}{15}, \frac{7}{30}, \frac{7}{6} \right). \end{aligned}$$

Hence

$$d(q, L) = d\left((1, \frac{3}{2}, \frac{3}{2}), \left(\frac{-7}{15}, \frac{7}{30}, \frac{7}{6}\right)\right) = \sqrt{\frac{58}{15}}.$$

30. Let $a, b \in \mathbb{R}^3$. Show that $|a \times b|^2 = |a|^2|b|^2 - \langle a, b \rangle^2$.

Solution.

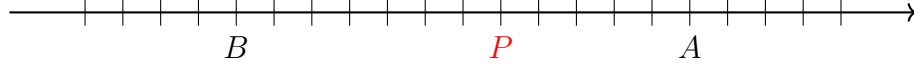
By definition:

$$\begin{aligned} a \times b &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\ \langle a, b \rangle &= a_1b_1 + a_2b_2 + a_3b_3. \end{aligned}$$

Brute force computation reduces both sides of the given equality to the expression:

$$a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2.$$

31. Let A, B be the two points indicated on the following Euclidean line:



and let λ_1, λ_2 be positive real numbers.

- (a) Show that if $\lambda_1 + \lambda_2 = 1$, then a linear combination of points $P = \lambda_1 A + \lambda_2 B$ does not depend on the location of the origin. Let O and O' be any two points of origin and let $P = \lambda_1 A + \lambda_2 B$.

Solution. A change of origin (in any dimension including 1) is expressed by Chasle's equation

$$\overrightarrow{OX} = \overrightarrow{OO'} + \overrightarrow{O'X}$$

In particular, $\overrightarrow{OA} = \overrightarrow{OO'} + \overrightarrow{O'A}$ and $\overrightarrow{OB} = \overrightarrow{OO'} + \overrightarrow{O'B}$. Hence,

$$\begin{aligned} O + \lambda_1 \overrightarrow{OA} + \lambda_2 \overrightarrow{OB} &= O + \lambda_1 \overrightarrow{OO'} + \lambda_1 \overrightarrow{O'A} + \lambda_2 \overrightarrow{OO'} + \lambda_2 \overrightarrow{O'B} \\ &= O + (\lambda_1 + \lambda_2) \overrightarrow{OO'} + \lambda_1 \overrightarrow{O'A} + \lambda_2 \overrightarrow{O'B} \\ &= O + \overrightarrow{OO'} + \lambda_1 \overrightarrow{O'A} + \lambda_2 \overrightarrow{O'B} \\ &= O' + \lambda_1 \overrightarrow{O'A} + \lambda_2 \overrightarrow{O'B} \end{aligned}$$

Both initial and final expressions describe the same point P .

- (b) Show that the ratio of the distances $PA : PB$ is $\lambda_2 : \lambda_1$. Hence locate the point $P = \frac{7}{12}A + \frac{5}{12}B$ on the above scale.

Solution.

$$\begin{aligned} d(P, A) &= |A - P| \\ &= |A - (\lambda_1 A + \lambda_2 B)| \\ &= |(1 - \lambda_1)A - \lambda_2 B| \\ &= |\lambda_2| \cdot |A - B| \\ &= \lambda_2 \cdot d(A, B). \end{aligned}$$

A similar argument shows that $d(P, B) = \lambda_1 d(A, B)$. Hence $\frac{d(P, A)}{d(P, B)} = \frac{\lambda_2}{\lambda_1}$. In particular if $P = \frac{7}{12}A + \frac{5}{12}B$ then $\frac{d(P, A)}{d(P, B)} = \frac{5}{7}$ (see diagram above).

This also provides a solution to part (a) since the ratio $\frac{d(P, A)}{d(P, B)}$ uniquely determines P .

- (c) Show that the point $Q = \frac{1}{12}A + \frac{7}{12}B$ cannot be located unless the origin is known.

Solution.

Counter example: If the origin O is the point A then Q is the point B . If the origin is B , then Q is the point A . Thus the location of Q depends on the origin.

This example shows that that formula for center of mass is independent of reference frame. According to relativity, the laws of science should be frame independent. In particular, in a relativistic model, there is no experiment which can distinguish between the earth going around the sun and the sun going around the earth.

32. Let P be a point and n a unit vector in $\mathbb{R}^3$. Let M, N be two distinct points in the plane π through P with normal n . Show that line through M and N is contained in the plane π .

Solution.

The plane π has equation $n \cdot (X - P) = 0$. Since $M, N \in \pi$

$$n \cdot (M - P) = 0$$

$$n \cdot (N - P) = 0$$

Let X be a point on the line containing M and N . Then $X = tM + (1 - t)N$ for some $t \in \mathbb{R}$. Then

$$\begin{aligned} n \cdot (X - P) &= n \cdot (tM + (1 - t)N - P) \\ &= tn \cdot M + (1 - t)n \cdot N - n \cdot P \\ &= tn \cdot P + (1 - t)n \cdot P - n \cdot P \\ &= 0 \end{aligned}$$

Hence $X \in \pi$ and so the line is contained in π .

33. (a) Write the line $2x - 4y + 5 = 0$ in the vector form $P + [u]$ where $P \in \mathbb{R}^2$ and $\vec{u}$ is a unit vector. **Solution.**

Let $P = (0, \frac{4}{5})$. Since $(2, -4)$ is normal to the line, $\vec{v} = (4, 2)$ is a direction vector. Take $u = \frac{\vec{v}}{\|\vec{v}\|} = (\frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}}) = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$.

- (b) Let $P = (4, 3)$ and $\vec{u} = (1, 2)$, write the image of the curve $\gamma(t) = P + tu$, $t \in \mathbb{R}$ in the form $y = mx + b$.

Solution.

$$\gamma(t) = (4, 3) + t(1, 2).$$

$$x = 4 + t$$

$$y = 3 + 2t$$

Eliminating t gives $y = 3 + 2(x - 4) = 2x - 5$.

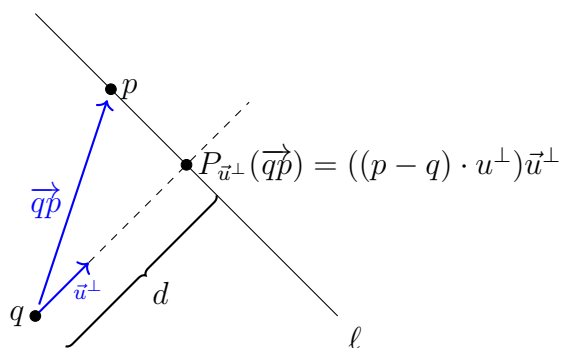
34. (a) Find a unit vector $u^\perp$ perpendicular to the line ℓ given by $ax + by + c = 0$ where a and b are not both zero.

Since $(a, b) \perp \ell$, we may take $u^\perp = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$.

- (b) Let p be the point $(0, -\frac{c}{b}) \in \ell$. and let $q = (\xi, \eta) \in \mathbb{R}^2$. Compute the projection $P_{u^\perp}(\vec{pq})$ of the free vector $\vec{pq}$.

Since $u^\perp$ is a unit vector,

$$\begin{aligned} P_{u^\perp}(\vec{pq}) &= \langle q - p, u^\perp \rangle u^\perp \\ &= \left\langle (\xi, \eta) - \left(0, -\frac{c}{b}\right), u^\perp \right\rangle u^\perp \\ &= \left\langle \left(\xi, \eta + \frac{c}{b}\right), \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right) \right\rangle u^\perp \\ &= \frac{a\xi + b\eta + c}{\sqrt{a^2 + b^2}} u^\perp. \end{aligned}$$



- (c) Let $d = |P_{u^\perp} \vec{pq}|$. What does d represent geometrically?

d is the perpendicular distance of the point $q = (\xi, \eta)$ to the line ℓ .

35. (a) Find a unit vector $\vec{n}$ perpendicular to the plane π given by $ax + by + cz + d = 0$ where a , b and c are not all zero.

Since $(a, b, c) \perp \pi$, we may take $n = \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right)$.

- (b) Let p, p' be distinct points in π . Show that p and p' have the same projections on n .

Solution.

If $p = (p_1, p_2, p_3) \in \pi$. Then

$$ap_1 + bp_2 + cp_3 + d = 0$$

It follows that

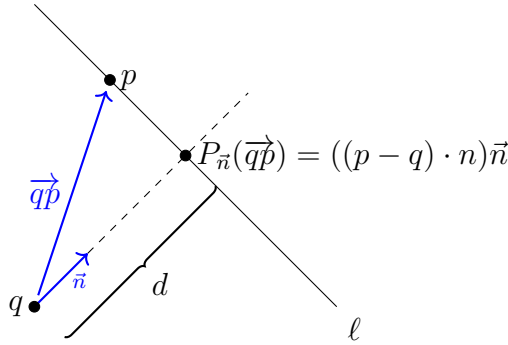
$$p \cdot n = \frac{-d}{\sqrt{a^2 + b^2 + c^2}}$$

$$\begin{aligned} \text{Proj}_n(p) &= (p \cdot n) \vec{n} \\ &= \frac{-d}{\sqrt{a^2 + b^2 + c^2}} \vec{n} \end{aligned}$$

Since this is independent of p it is the same for all points $p \in \pi$.

- (c) Let $q = (\xi, \eta, \zeta) \in \mathbb{R}^3$. Compute (in terms of a, b, c, d) the projection $P_n(\vec{pq})$ of the free vector $\vec{pq}$ on $\vec{n}$.

$$\begin{aligned}
 P_n(\vec{pq}) &= (q - p) \cdot n \\
 &= (q \cdot n - p \cdot n)n \\
 &= \left(\frac{a\xi + b\eta + c\zeta}{\sqrt{a^2 + b^2 + c^2}} - \frac{-d}{\sqrt{a^2 + b^2 + c^2}} \right) \vec{n} \\
 &= \left(\frac{a\xi + b\eta + c\zeta + d}{\sqrt{a^2 + b^2 + c^2}} \right) \vec{n}
 \end{aligned}$$



- (d) What does $|P_n \vec{pq}|$ represent geometrically?

d is the perpendicular distance of the point $q = (\xi, \eta, \zeta)$ to the plane π .

36. Let $a, b, c \in \mathbb{R}^2$. If $b \wedge c = b_1 c_2 - b_2 c_1$ is the oriented area of the parallelogram determined by b, c and $a^\perp = (a_2, -a_1)$ Show that

$$(a \cdot c)b - (a \cdot b)c = (b \wedge c) a^\perp.$$

Solution.

$$\begin{aligned}
 (a \cdot c)b - (a \cdot b)c &= (a_1 c_1 + a_2 c_2)b - (a_1 b_1 + a_2 b_2)c \\
 &= (a_1 c_1 + a_2 c_2)(b_1, b_2) - (a_1 b_1 + a_2 b_2)(c_1, c_2) \\
 &= (a_1 b_1 c_1 + a_2 b_1 c_2 - a_1 b_1 c_1 - a_2 b_2 c_1, a_1 b_2 c_1 + a_2 b_2 c_2 - a_1 b_1 c_2 - a_2 b_2 c_2) \\
 &= (a_2 b_1 c_2 - a_2 b_2 c_1, a_1 b_2 c_1 - a_1 b_1 c_2) \\
 &= (a_2, -a_1)(b \wedge c) \\
 &= (a_2(b \wedge c), -a_1(b \wedge c)) \\
 &= (b \wedge c) a^\perp
 \end{aligned}$$

37. Find a unit vector u parallel to the line of intersection of the planes: $2x - y + z = 1$,

and $3x + y + z = 2$.

$$\begin{aligned}n_1 &= (2, -1, 1) \\n_2 &= (3, 1, 1) \\n_1 \times n_2 &= (-2, 1, 5)\end{aligned}$$

Hence $u = \frac{1}{\sqrt{30}}(-2, 1, 5)$

38. Let P be a point on the plane $2x + 3y + 4z = 5$. Find the length of the projection of P on the normal vector to the plane.

Solution.

Let $n = (2, 3, 4)$ and $\hat{n} = \frac{n}{\|n\|} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$ and Let $P = (\xi, \eta, \zeta)$ be a point on the plane

$$\begin{aligned}\text{Proj}_n(P) &= (P \cdot \hat{n}) \hat{n} \\&= \left(\frac{2\xi + 3\eta + 4\zeta}{\sqrt{29}}\right) \hat{n} \\&= \frac{5}{\sqrt{29}} \hat{n}\end{aligned}$$