

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

EXERCISES ON SURFACE INTEGRALS

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1. Discuss briefly the geometric motivation underlying the definition of the integral of a scalar function over a smooth parametrized surface.

Solution. Let $S = X(U)$ be a smooth parametrized surface. A small parameter rectangle $[u, u + \Delta u] \times [v, v + \Delta v]$ is sent to a curved patch which is approximated by the parallelogram spanned by $X_u \Delta u$ and $X_v \Delta v$. That parallelogram has area $\|X_u \times X_v\| \Delta u \Delta v$. If f is continuous, the sum of the values $f(X(u, v))$ times these areas tends to

$$\int_S f d\sigma = \iint_U f(X(u, v)) \|X_u \times X_v\| du dv.$$

When $f \equiv 1$ this is surface area; when f is a density it is mass.

2. Show that the surface area of a sphere S of radius ρ is $4\pi\rho^2$ by calculating $\int_S 1 d\sigma$.

Solution. Use spherical coordinates,

$$X(\phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi),$$

with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Then

$$X_\phi \times X_\theta = \rho^2 \sin \phi (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi), \quad \|X_\phi \times X_\theta\| = \rho^2 \sin \phi.$$

Hence

$$\int_S 1 d\sigma = \int_0^\pi \int_0^{2\pi} \rho^2 \sin \phi d\theta d\phi = \rho^2 \cdot 2\pi \int_0^\pi \sin \phi d\phi = 4\pi\rho^2.$$

3. Evaluate $\int_S y d\sigma$, where S is given by $z = x + y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

Solution.

$$\begin{aligned} X(u, v) &= (u, v, u + v^2), \\ X_u &= (1, 0, 1), \quad X_v = (0, 1, 2v), \\ X_u \times X_v &= (-1, -2v, 1), \\ \|X_u \times X_v\| &= \sqrt{2 + 4v^2} = \sqrt{2} \sqrt{1 + 2v^2}. \end{aligned}$$

Thus

$$\int_S y d\sigma = \sqrt{2} \int_0^2 \int_0^1 v \sqrt{1 + 2v^2} du dv = \sqrt{2} \int_0^2 v \sqrt{1 + 2v^2} dv.$$

The substitution $w = 1 + 2v^2$, $dw = 4v dv$, sends $v = 0$ to $w = 1$ and $v = 2$ to $w = 9$, so

$$\int_S y d\sigma = \frac{\sqrt{2}}{4} \int_1^9 w^{1/2} dw = \frac{\sqrt{2}}{6} \left[w^{3/2} \right]_1^9 = \frac{\sqrt{2}}{6} (27 - 1) = \frac{13}{3} \sqrt{2}.$$

4. Evaluate $\int_S x^2 d\sigma$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution.

$$\begin{aligned} X(\phi, \theta) &= (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi), \\ X_\phi &= (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi), \\ X_\theta &= (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0), \\ X_\phi \times X_\theta &= (\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi), \\ \|X_\phi \times X_\theta\| &= \sin \phi \quad (0 \leq \phi \leq \pi). \end{aligned}$$

(The last identity holds because $\|X_\phi \times X_\theta\|^2 = \sin^4 \phi + \sin^2 \phi \cos^2 \phi = \sin^2 \phi$.) Therefore

$$\int_S x^2 d\sigma = \int_0^\pi \int_0^{2\pi} (\sin \phi \cos \theta)^2 \sin \phi d\theta d\phi = \left(\int_0^\pi \sin^3 \phi d\phi \right) \left(\int_0^{2\pi} \cos^2 \theta d\theta \right).$$

The θ -integral equals π . For the ϕ -integral, $\sin^3 \phi = (1 - \cos^2 \phi) \sin \phi$, and $w = \cos \phi$ gives

$$\int_0^\pi \sin^3 \phi d\phi = \int_{-1}^1 (1 - w^2) dw = \frac{4}{3}.$$

Hence the surface integral is $\frac{4\pi}{3}$.

5. Evaluate $\int_S yz d\sigma$, where S is the part of the plane $z = y + 3$ inside the cylinder $x^2 + y^2 = 1$.

Solution.

$$\begin{aligned} X(u, v) &= (u, v, v + 3), \\ X_u &= (1, 0, 0), \quad X_v = (0, 1, 1), \\ X_u \times X_v &= (0, -1, 1), \\ \|X_u \times X_v\| &= \sqrt{2}. \end{aligned}$$

The parameter domain is the disk $D : u^2 + v^2 \leq 1$, so

$$\int_S yz d\sigma = \sqrt{2} \iint_D v(v + 3) du dv = \sqrt{2} \iint_D (v^2 + 3v) du dv.$$

The term $3v$ is odd in v and integrates to 0 over D . In polar coordinates,

$$\iint_D v^2 du dv = \int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta \cdot r dr d\theta = \frac{1}{4} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{\pi}{4}.$$

Therefore

$$\int_S yz \, d\sigma = \sqrt{2} \cdot \frac{\pi}{4} = \frac{\pi\sqrt{2}}{4}.$$

6. Calculate the mass of the part of the plane $3x + 2y + z = 6$ in the first octant if the surface density is $\rho(x, y, z) = y$.

Solution. A parametrization of the first-octant portion is

$$\begin{aligned} X(u, v) &= (u, v, 6 - 3u - 2v), & 0 \leq u \leq 2, & \quad 0 \leq v \leq 3 - \frac{3}{2}u, \\ X_u &= (1, 0, -3), & X_v &= (0, 1, -2), \\ X_u \times X_v &= (3, 2, 1), \\ \|X_u \times X_v\| &= \sqrt{14}. \end{aligned}$$

The mass is

$$\int_S y \, d\sigma = \sqrt{14} \int_0^2 \int_0^{3-3u/2} v \, dv \, du = \sqrt{14} \int_0^2 \frac{1}{2} \left(3 - \frac{3u}{2}\right)^2 du.$$

The substitution $w = 3 - 3u/2$ yields

$$\int_S y \, d\sigma = \sqrt{14} \cdot \frac{1}{3} \int_0^3 w^2 \, dw = 3\sqrt{14}.$$

7. Calculate the mass of the triangle with vertices $(1, 0, 0)$, $(1, 1, 1)$ and $(0, 0, 2)$ if the surface density is $\rho(x, y, z) = xz$.

Solution. The unique plane through the three points is $z = 2 - 2x + y$. The projection onto the xy -plane is the triangle with vertices $(1, 0)$, $(1, 1)$ and $(0, 0)$, i.e. $0 \leq x \leq 1$, $0 \leq y \leq x$. Thus

$$\begin{aligned} X(u, v) &= (u, v, 2 - 2u + v), & 0 \leq u \leq 1, & \quad 0 \leq v \leq u, \\ X_u &= (1, 0, -2), & X_v &= (0, 1, 1), \\ X_u \times X_v &= (2, -1, 1), \\ \|X_u \times X_v\| &= \sqrt{6}. \end{aligned}$$

The mass is

$$\int_S xz \, d\sigma = \sqrt{6} \int_0^1 \int_0^u u(2 - 2u + v) \, dv \, du = \sqrt{6} \int_0^1 \left(2u^2 - \frac{3}{2}u^3\right) du = \frac{7}{24}\sqrt{6}.$$

8. Find the flux of $\mathbf{F}(x, y, z) = (y, x, z)$ through the surface S with parametrization

$$X(u, v) = (\cos v, \sin v, u), \quad 0 \leq u \leq 2, \quad \frac{\pi}{2} \leq v \leq \pi.$$

Solution.

$$\begin{aligned} X_u &= (0, 0, 1), & X_v &= (-\sin v, \cos v, 0), \\ X_u \times X_v &= (-\cos v, -\sin v, 0). \end{aligned}$$

On the surface, $\mathbf{F}(X(u, v)) = (\sin v, \cos v, u)$, so

$$\mathbf{F}(X) \cdot (X_u \times X_v) = -\sin v \cos v - \cos v \sin v = -2 \sin v \cos v = -\sin 2v.$$

Hence

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_{\pi/2}^{\pi} (-\sin 2v) dv du = \left(\int_0^2 du \right) \left[\frac{1}{2} \cos 2v \right]_{\pi/2}^{\pi} \\ &= 2 \cdot \frac{1}{2} (\cos 2\pi - \cos \pi) = 1 - (-1) = 2. \end{aligned}$$

(The chosen normal $X_u \times X_v = -(\cos v, \sin v, 0)$ points toward the z -axis.)

9. (a) Let $p \in \mathbb{R}^3$ and let $\vec{\alpha}, \vec{\beta}$ be orthonormal vectors in $\mathbb{R}^3$. Define $X(u, v) = p + u\vec{\alpha} + v\vec{\beta}$.
- Describe the image S of X . Show that X is a bijection onto its image.
 - Find the inverse map X^{-1} .

Solution. The image S is the affine plane through p spanned by $\vec{\alpha}$ and $\vec{\beta}$. If $X(u, v) = X(u', v')$, then $(u - u')\vec{\alpha} + (v - v')\vec{\beta} = \mathbf{0}$. Linear independence forces $u = u'$ and $v = v'$, so X is injective, hence bijective onto S . Because $\{\vec{\alpha}, \vec{\beta}\}$ is orthonormal,

$$X^{-1}(x, y, z) = ((\mathbf{r} - p) \cdot \vec{\alpha}, (\mathbf{r} - p) \cdot \vec{\beta}), \quad \mathbf{r} = (x, y, z).$$

In these coordinates $\|X_u \times X_v\| = \|\vec{\alpha} \times \vec{\beta}\| = 1$, so $d\sigma = du dv$.

- (b) Discuss briefly the definition of the integral of a scalar function over a smooth parametrized surface.

Solution. If $X : U \rightarrow \mathbb{R}^3$ is a smooth parametrization of S and f is continuous on S ,

$$\int_S f d\sigma = \iint_U f(X(u, v)) \|X_u \times X_v\| du dv.$$

The factor $\|X_u \times X_v\|$ converts parameter area into surface area. The value is independent of the choice of (orientation-preserving, regular) parametrization.

- (c) Show that the surface area of a sphere of radius ρ is $4\pi\rho^2$ by calculating $\int_S 1 d\sigma$.

Solution. As in item 2, $\|X_\phi \times X_\theta\| = \rho^2 \sin \phi$ on $[0, \pi] \times [0, 2\pi]$, and

$$\int_S 1 d\sigma = \int_0^\pi \int_0^{2\pi} \rho^2 \sin \phi d\theta d\phi = 4\pi\rho^2.$$