

Stokes's Theorem

Philip Pennance¹ Version: – February 2019.

1. Compute the surface integral $\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ where $\mathbf{F}$ is the vector field

$$\mathbf{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$$

and S is the face of the triangle with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$. Take n to be the unit normal in the direction $PQ \times PR$.

Solution.

Let Ω be the triangular region of the (u, v) plane

$$\Omega = \{(u, v) : 0 \leq u + v \leq 1, \quad u, v \geq 0\}.$$

The surface S has parametrization $X : \Omega \rightarrow \mathbb{R}^3$ given by

$$\begin{aligned} X(u, v) &= P + u(Q - P) + v(R - P) \\ &= P + u\overrightarrow{PQ} + v\overrightarrow{PR} \\ &= (1, 0, 0) + u(-1, 1, 0) + v(-1, 0, 1), \\ &= (1 - u - v, u, v), \quad 0 \leq u + v \leq 1, \quad u, v \geq 0. \end{aligned}$$

$$X_u = (-1, 1, 0).$$

$$X_v = (-1, 0, 1).$$

$$X_u \times X_v = (1, 1, 1).$$

$$\begin{aligned} \nabla \times F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= (-2z, -2x, -2y) \\ &= -2(z, x, y). \end{aligned}$$

$$\nabla \times F(X(u, v)) = -2(v, 1 - u - v, u).$$

$$\nabla \times F(X(u, v)) \cdot (X_u \times X_v) = -2.$$

$$\begin{aligned} \int \int_S \nabla \times \mathbf{F} \cdot \mathbf{n} ds &= \int \int_{\Omega} \nabla \times \mathbf{F}(X(u, v)) \cdot (X_u \times X_v) du dv \\ &= -2 \int_0^1 \int_0^{1-u} 1 dv du \\ &= -2 \int_0^1 1 - u du \\ &= -1. \end{aligned}$$

¹<https://pennance.us>

2. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$$

and let C be the triangle with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$. Verify Stoke's Theorem, in this case, by directly computing the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ and comparing the result with that of the previous question.

Solution.

$$\begin{array}{lll} \gamma_1(t) = P + t(Q - P) & \gamma_2(t) = Q + t(R - Q) & \gamma_3(t) = R + t(P - R) \\ = (1, 0, 0) + t(-1, 1, 0) & = (0, 1, 0) + t(0, -1, 1) & = (0, 0, 1) + t(1, 0, -1) \\ = (1 - t, t, 0) & = (0, 1 - t, t) & = (t, 0, 1 - t) \\ \gamma_1'(t) = (-1, 1, 0) & \gamma_2'(t) = (0, -1, 1) & \gamma_3'(t) = (1, 0, -1) \end{array}$$

Hence,

$$\begin{aligned} \int_{\gamma_1} \mathbf{F} &= \int_0^1 \mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) dt \\ &= \int_0^1 (1 - t + t^2, t, (1 - t)^2) \cdot (-1, 1, 0) dt \\ &= \int_0^1 -t^2 + 2t - 1 dt \\ &= -\frac{1}{3}. \end{aligned}$$

Similarly,

$$\int_{\gamma_2} \mathbf{F} = \int_{\gamma_3} \mathbf{F} = -\frac{1}{3}.$$

It follows that

$$\int_C \mathbf{F} = \int_{\gamma_1} \mathbf{F} + \int_{\gamma_2} \mathbf{F} + \int_{\gamma_3} \mathbf{F} = -1.$$

3. Give four examples which illustrate that microscopic circulation may not resemble macroscopic. Solution.

(a)

$$\begin{aligned} F(x, y, z) &= \frac{(-y, x, 0)}{x^2 + y^2} \\ \nabla \times F(x, y, z) &= (0, 0, 0) \end{aligned}$$

(b)

$$\begin{aligned} F(x, y, z) &= (-y, x, 0) \\ \nabla \times F(x, y, z) &= (0, 0, 2) \end{aligned}$$

(c)

$$F(x, y, z) = \frac{(-y, x, 0)}{(x^2 + y^2)^{3/2}}$$

$$\nabla \times F = \frac{(0, 0, -1)}{(x^2 + y^2)^{3/2}}.$$

(d)

$$F(x, y, z) = (y, 0, 0)$$

$$\nabla \times F = (0, 0, -1).$$

4. Let S be the part of the sphere

$$x^2 + y^2 + z^2 = 4$$

that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy -plane. Use Stokes' theorem, to evaluate

$$\iint_S \nabla \times F \cdot d\vec{\sigma},$$

where

$$F = xz\hat{i} + yz\hat{j} + xy\hat{k},$$

Solution.

The boundary of S is easily found to be the circle given by

$$x^2 + y^2 = 1, \quad z = \sqrt{3}.$$

Looking down along the positive z -axis this should be oriented counterclockwise. Hence let

$$\gamma(t) = (\cos t, \sin t, \sqrt{3}), 0 \leq t \leq 2\pi.$$

$$\gamma'(t) = (-\sin t, \cos t, 0)$$

$$F(\gamma(t)) = (\sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t),$$

$$\begin{aligned} \int_S \nabla \times F \cdot n \, d\sigma &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= 0. \end{aligned}$$

5. Let C be the unit circle oriented counter-clockwise. Evaluate

$$\int_C x^2 z \, dx + 3x \, dy - y^3 \, dz$$