

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

EXERCISES ON LINE INTEGRALS

Prof. Philip Pennance

1. Let $f(x, y) = x + y$ and let C be the curve defined by

$$\gamma(t) = (2 \cos t, 2 \sin t), \quad 0 \leq t \leq \frac{\pi}{2}.$$

Compute the line integral $\int_C f(x, y) ds$.

Solution.

$$\begin{aligned}\gamma'(t) &= (-2 \sin t, 2 \cos t), \\ ds &= \|\gamma'(t)\| dt = 2 dt, \\ \int_C (x + y) ds &= \int_0^{\pi/2} (2 \cos t + 2 \sin t) \cdot 2 dt = 8.\end{aligned}$$

2. Let $f(x, y) = x^2 y$ and let C be the segment from $(3, 2)$ to $(7, 5)$. Compute $\int_C f(x, y) ds$.

Solution.

$$\begin{aligned}\gamma(t) &= (3, 2) + t(4, 3), \quad 0 \leq t \leq 1, \\ x(t) &= 3 + 4t, \quad y(t) = 2 + 3t, \\ \gamma'(t) &= (4, 3), \quad \|\gamma'(t)\| = 5, \\ ds &= 5 dt, \\ \int_C x^2 y ds &= \int_0^1 (3 + 4t)^2 (2 + 3t) \cdot 5 dt = \frac{3065}{6}.\end{aligned}$$

3. Let $f(x, y) = x^2 y$ and let C be the segment from $(3, 2)$ to $(7, 5)$. Compute $\int_C x^2 y dx$.

Solution. $\int_C x^2 y dx$ is the line integral of the vector field $\mathbf{F}(x, y) = (x^2 y, 0)$. Hence, with the same parametrization as above,

$$\begin{aligned}\int_C x^2 y dx &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_0^1 ((3 + 4t)^2 (2 + 3t), 0) \cdot (4, 3) dt \\ &= \int_0^1 4(3 + 4t)^2 (2 + 3t) dt = \frac{1226}{3}.\end{aligned}$$

4. Let $f(x, y) = x$ and let C be the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$. Compute $\int_C f(x, y) ds$.

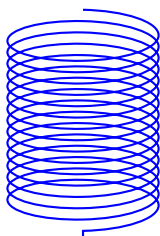
Solution.

$$\begin{aligned}\int_C f(x, y) ds &= \int_0^1 f(x, x^2) \sqrt{1 + (2x)^2} dx \\ &= \int_0^1 x \sqrt{1 + 4x^2} dx = \frac{1}{12} (5\sqrt{5} - 1).\end{aligned}$$

5. The wire in a solenoid is non-uniform, with mass density $\lambda(x, y, z) = kz$, where k is a constitutive constant. The wire is the image of

$$\gamma(t) = (2 \cos t, 2 \sin t, 3t), \quad 0 \leq t \leq 20\pi.$$

Find the total mass of the solenoid.



Solution. Let M be the mass of the wire.

$$\begin{aligned}\gamma'(t) &= (-2 \sin t, 2 \cos t, 3), \\ \|\gamma'(t)\| &= \sqrt{13}, \\ M &= \int_{\gamma} \lambda ds = \int_0^{20\pi} \lambda(\gamma(t)) \|\gamma'(t)\| dt \\ &= \int_0^{20\pi} (3kt) \sqrt{13} dt = 3\sqrt{13} k \int_0^{20\pi} t dt = 600\sqrt{13} k \pi^2.\end{aligned}$$

6. Let $\mathbf{F}(x, y) = (x, y^2)$. Calculate $\int_{\Gamma} \mathbf{F} \cdot d\mathbf{s}$, where Γ is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(1, 0)$, oriented clockwise.

Solution. Here $\partial F_2 / \partial x - \partial F_1 / \partial y = 0$. The domain $\mathbb{R}^2$ is simply connected and $\mathbf{F}$ is C^1 , so $\mathbf{F}$ is conservative. A potential is

$$\phi(x, y) = \frac{x^2}{2} + \frac{y^3}{3}.$$

The integral of $\mathbf{F}$ around any closed curve is therefore 0.

7. Let $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $\mathbf{F}(x, y) = (2xy, x^2 + 3y^2)$. Find a function ϕ such that $\mathbf{F} = \nabla\phi$.

Solution.

$$\frac{\partial\phi}{\partial x} = 2xy, \quad (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 + 3y^2. \quad (2)$$

Integrating (1) with respect to x gives $\phi(x, y) = x^2y + \psi(y)$. Differentiating with respect to y and comparing with (2) yields $\psi'(y) = 3y^2$, so $\psi(y) = y^3 + C$. Taking $C = 0$ gives

$$\phi(x, y) = x^2y + y^3.$$

8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where

$$\mathbf{F}(x, y, z) = (x^2 + y, -7yz, 2xz^2)$$

and C is the path from $(0, 0, 0)$ to $(1, 1, 1)$ given by $\gamma(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$.

Solution.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_0^1 (2t^2, -7t^5, 2t^7) \cdot (1, 2t, 3t^2) dt \\ &= \int_0^1 (2t^2 - 14t^6 + 6t^9) dt = -\frac{11}{15}. \end{aligned}$$

9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y) = (y, x^2 + y^2)$ and C is the path from $(-2, 0)$ to $(2, 0)$ given by $\gamma(t) = (t, \sqrt{4 - t^2})$, $-2 \leq t \leq 2$.

Solution.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{-2}^2 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_{-2}^2 (\sqrt{4 - t^2}, 4) \cdot \left(1, -\frac{t}{\sqrt{4 - t^2}}\right) dt \\ &= \int_{-2}^2 \left(\sqrt{4 - t^2} - \frac{4t}{\sqrt{4 - t^2}}\right) dt = 2\pi. \end{aligned}$$

(The first integrand is the upper semicircle of radius 2; the second is odd.)

10. Evaluate $\int_{\gamma} x^3 y^2 z \, dz$ where $x = 2t$, $y = t^2$, $z = t^2$, $0 \leq t \leq 1$.

Solution. Let $\mathbf{F}(x, y, z) = (0, 0, x^3 y^2 z)$ and $\gamma(t) = (2t, t^2, t^2)$, $0 \leq t \leq 1$.

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt \\ &= \int_0^1 (0, 0, 8t^9) \cdot (2, 2t, 2t) \, dt \\ &= \int_0^1 16t^{10} \, dt = \frac{16}{11}. \end{aligned}$$

11. Evaluate $\int_C \sqrt{y} \, dx + \sqrt{x} \, dy$, where C is the boundary of the region bounded by $x = 0$, $y = 1$ and $y = x^2$, traversed once counterclockwise.

Solution. Write $C = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned} \gamma_1(t) &= (t, t^2), & 0 \leq t \leq 1, \\ \gamma_2(t) &= (1 - t, 1), & 0 \leq t \leq 1, \\ \gamma_3(t) &= (0, 1 - t), & 0 \leq t \leq 1. \end{aligned}$$

Let $\mathbf{F}(x, y) = (\sqrt{y}, \sqrt{x})$. Then

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (\sqrt{t^2} \cdot 1 + \sqrt{t} \cdot 2t) \, dt = \frac{13}{10}, \\ \int_{C_2} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (\sqrt{1} \cdot (-1) + \sqrt{1-t} \cdot 0) \, dt = -1, \\ \int_{C_3} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (\sqrt{1-t} \cdot 0 + \sqrt{0} \cdot (-1)) \, dt = 0. \end{aligned}$$

Hence

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \frac{13}{10} - 1 + 0 = \frac{3}{10}.$$

12. Determine the work done by $\mathbf{F}(x, y) = (x, y + 2)$ in moving a particle along the cycloid

$$\gamma(t) = (t - \sin t, 1 - \cos t), \quad 0 \leq t \leq 2\pi.$$

Solution. A potential is $\phi(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + 2y$. Thus

$$W = \phi(\gamma(2\pi)) - \phi(\gamma(0)) = \phi(2\pi, 0) - \phi(0, 0) = 2\pi^2.$$

13. (a) Show that $\mathbf{F}(x, y) = (x^2 y, y^3)$ is not conservative.

- (b) Calculate the integral of $\mathbf{F}$ along $\gamma(t) = (t, t)$, $t \in [0, 1]$.
- (c) Write a parametrization of the reverse path γ^r .
- (d) Let A be a point on the sphere $x^2 + y^2 + z^2 = 1$ and B a point on the sphere $x^2 + y^2 + z^2 = 4$, and let $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ be any path from A to B . Calculate

$$\int_{\gamma} \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2}.$$

Solution.

- (a) $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = x^2 \neq 0$, so $\mathbf{F}$ is not conservative.
- (b) $\gamma'(t) = (1, 1)$ and

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t^3, t^3) \cdot (1, 1) dt = \int_0^1 2t^3 dt = \frac{1}{2}.$$

- (c) $\gamma^r(t) = \gamma(1 - t) = (1 - t, 1 - t)$.
- (d) The integrand is $d(\frac{1}{2} \log(x^2 + y^2 + z^2))$. On the two spheres the radii are 1 and 2, so the integral equals $\log 2$, independently of the path (so long as it avoids the origin).

14. Let $\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ and let $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ satisfy $\gamma(0) = (1, 1, 1)$ and $\gamma(1) = (2, 2, 2)$. Evaluate

- (a) $\nabla \phi(x, y, z)$;
- (b) $\int_{\gamma} \nabla \phi \cdot d\mathbf{r}$.

Solution.

- (a) $\nabla \phi(x, y, z) = -\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$.
- (b) Since the integrand is a gradient,

$$\int_{\gamma} \nabla \phi \cdot d\mathbf{r} = \phi(2, 2, 2) - \phi(1, 1, 1) = \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{1}{2\sqrt{3}}.$$

15. Let

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

- (a) Find the integral of $\mathbf{F}$ around the circle $x^2 + y^2 = 4$, oriented counterclockwise.

Solution. Parametrize by $\gamma(t) = (2 \cos t, 2 \sin t)$, $0 \leq t \leq 2\pi$. Then

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \left(-\frac{\sin t}{2}, \frac{\cos t}{2} \right) \cdot (-2 \sin t, 2 \cos t) dt = 2\pi.$$

- (b) Let F_1 and F_2 be the components of $\mathbf{F}$.

i. Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Solution. For $(x, y) \neq (0, 0)$,

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_1}{\partial y},$$

so the difference is 0.

- ii. Are F_1 and F_2 continuous on $\mathbb{R}^2$?

Solution. No. They are undefined at the origin, so the natural domain $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not simply connected.

- iii. Is $\mathbf{F}$ conservative? Explain.

Solution. No: the integral around a closed curve is nonzero, so $\mathbf{F}$ admits no single-valued potential on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

16. Calculate $\nabla \arctan(y/x)$.

Solution. Where $x \neq 0$,

$$\begin{aligned} \frac{\partial}{\partial x} \arctan \frac{y}{x} &= \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}, \\ \frac{\partial}{\partial y} \arctan \frac{y}{x} &= \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Hence, on $\{x > 0\}$ (where $\arctan(y/x)$ coincides with the polar angle),

$$\nabla \arctan \frac{y}{x} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

This identity does *not* give a single-valued potential on a full punctured neighbourhood of the origin.

17. Suppose f_1 and f_2 are continuously differentiable functions $\mathbb{R} \rightarrow \mathbb{R}$. Explain why $\mathbf{F}(x, y) = (f_1(x), f_2(y))$ is conservative.

Solution.

$$\phi(x, y) = \int f_1(x) dx + \int f_2(y) dy$$

is a potential: $\nabla\phi = \mathbf{F}$.

18. Let

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$. Is $\mathbf{F}$ conservative? Explain.

Solution. The combination of partial derivatives vanishes on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Nevertheless $\mathbf{F}$ is not conservative on that domain, because its integral around the unit circle is $2\pi \neq 0$. (Note that these components are *not* of the separated form $(f_1(x), f_2(y))$ from the previous item.)

19. Evaluate $\int_C \frac{-y dx + x dy}{x^2 + y^2}$ where C is the unit circle, oriented clockwise.

Solution. The parametrization $\gamma(t) = (\cos(-t), \sin(-t))$, $0 \leq t \leq 2\pi$, traverses the circle clockwise. Then

$$dx = \sin(-t) dt, \quad dy = -\cos(-t) dt,$$

and

$$\int_{\gamma} \frac{-y dx + x dy}{x^2 + y^2} = \int_0^{2\pi} \left(-\sin^2(-t) - \cos^2(-t) \right) dt = -\int_0^{2\pi} dt = -2\pi.$$

20. Evaluate $\int_C \frac{-y dx + x dy}{x^2 + y^2}$ where C is the circle $(x - 2)^2 + y^2 = 1$, oriented clockwise.

Solution. This circle lies in the half-plane $x \geq 1 > 0$, on which

$$\phi(x, y) = \arctan \frac{y}{x}$$

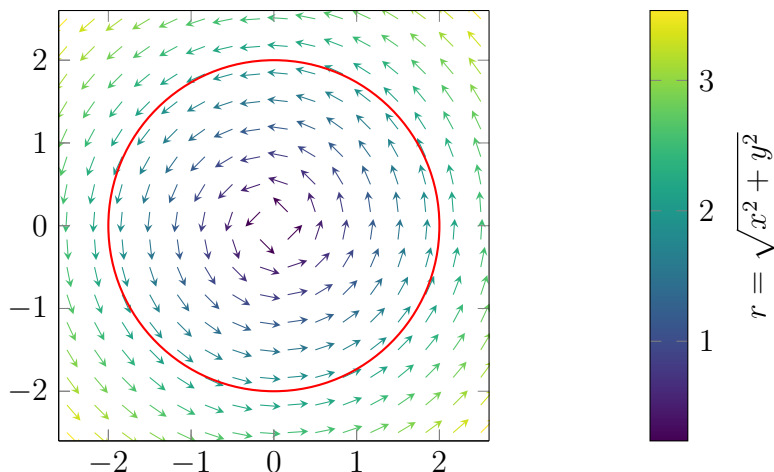
is a C^1 potential. The integral around a closed curve is therefore 0.

21. By evaluating the integral of

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

around the circle $x^2 + y^2 = 4$, show that $\mathbf{F}$ is not conservative on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Vector field $\mathbf{F}(x, y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$



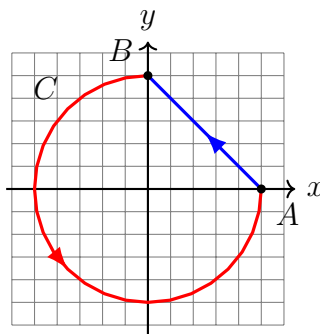
Solution. The computation in item 15 shows that the circulation is $2\pi \neq 0$. A conservative field would have vanishing circulation about every closed curve in the domain. Hence $\mathbf{F}$ is not conservative on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

22. Find the integral of

$$\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right)$$

around the curve C consisting of the straight line from $A = (5, 0)$ to $B = (0, 5)$, followed by the longer arc of the circle of radius 5 centred at the origin, from B back to A .

Answer: -2π .



Verification by direct integration. The segment is $\gamma_1(t) = 5(1 - t, t)$, $0 \leq t \leq 1$, so $\gamma_1'(t) = (-5, 5)$. With $\Delta(t) = 2t^2 - 2t + 1$,

$$\begin{aligned} \mathbf{F}(\gamma_1(t)) &= \left(\frac{t}{5\Delta(t)}, -\frac{1-t}{5\Delta(t)} \right), \\ \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left(\frac{t}{5\Delta(t)} \cdot (-5) - \frac{1-t}{5\Delta(t)} \cdot 5 \right) dt = \int_0^1 \frac{-1}{\Delta(t)} dt = -\frac{\pi}{2}. \end{aligned}$$

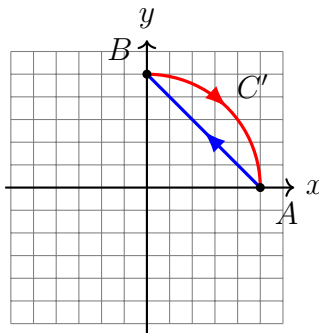
The longer arc is $\gamma_2(s) = (5 \cos s, 5 \sin s)$, $\pi/2 \leq s \leq 2\pi$. Then $\mathbf{F}(\gamma_2(s)) \cdot \gamma_2'(s) = -1$, and

$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\pi/2}^{2\pi} (-1) ds = -\frac{3\pi}{2}.$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\frac{\pi}{2} - \frac{3\pi}{2} = -2\pi.$$

23. Find the integral of the same field around the curve C' consisting of the straight line from $A = (5, 0)$ to $B = (0, 5)$, followed by the shorter arc of the same circle from B back to A .



Verification by direct integration. The line integral along γ_1 is again $-\pi/2$. The shorter arc may be written $\gamma_2(s) = (5 \sin s, 5 \cos s)$, $0 \leq s \leq \pi/2$, running from B to A . Then $\mathbf{F}(\gamma_2(s)) \cdot \gamma_2'(s) = 1$, and

$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} 1 ds = \frac{\pi}{2}.$$

Hence

$$\int_{C'} \mathbf{F} \cdot d\mathbf{s} = -\frac{\pi}{2} + \frac{\pi}{2} = 0.$$

Alternative solution using differential forms. Let

$$\alpha = \frac{y dx - x dy}{x^2 + y^2}.$$

In polar coordinates, $y dx - x dy = -r^2 d\theta$, so $\alpha = -d\theta$. Along C , θ increases by 2π , and $\int_C \alpha = -2\pi$. Along C' , θ returns to its initial value, and the integral is 0.

24. Find $\int_C \mathbf{F} \cdot d\mathbf{s}$ where

$$\mathbf{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$$

and $C = PQ \cup QR \cup RP$ is the boundary of the triangle with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$.

$$\begin{aligned}\gamma_1(t) &= (1-t, t, 0), & \gamma_2(t) &= (0, 1-t, t), & \gamma_3(t) &= (t, 0, 1-t), \\ \gamma_1'(t) &= (-1, 1, 0), & \gamma_2'(t) &= (0, -1, 1), & \gamma_3'(t) &= (1, 0, -1).\end{aligned}$$

Solution.

$$\begin{aligned}\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (1-t+t^2, t, (1-t)^2) \cdot (-1, 1, 0) dt \\ &= \int_0^1 (-1+2t-t^2) dt = -\frac{1}{3}.\end{aligned}$$

The integrals along γ_2 and γ_3 each equal $-\frac{1}{3}$ as well. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -1.$$

25. Let C be the line segment directed from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$. Show that

$$\int_C x dy - y dx = x_1 y_2 - x_2 y_1.$$

What is the geometric meaning of this integral?

Solution. The left-hand side is the line integral of $\mathbf{F}(x, y) = (-y, x)$. Parametrize

$$\gamma(t) = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2), \quad 0 \leq t \leq 1,$$

so $\gamma'(t) = (x_2 - x_1, y_2 - y_1)$. Then

$$\begin{aligned}\int_C x dy - y dx &= \int_0^1 (-(1-t)y_1 - ty_2, (1-t)x_1 + tx_2) \cdot (x_2 - x_1, y_2 - y_1) dt \\ &= \int_0^1 (x_1 y_2 - x_2 y_1) dt = x_1 y_2 - x_2 y_1,\end{aligned}$$

which is the signed area of the parallelogram with vertices O , P , Q and $P+Q$. Together with Green's theorem this yields the usual polygonal area formula.

26. Let $\mathbf{F}(x, y) = (x^2 y, y^3)$ and $\gamma(t) = (t, 2t)$, $t \in [0, 1]$.

- (a) Show that $\mathbf{F}$ is not conservative.

Solution. $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} = x^2 \neq 0.$

- (b) Calculate the integral of $\mathbf{F}$ along γ .

Solution.

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t^3, 8t^3) \cdot (1, 2) dt = \int_0^1 18t^3 dt = \frac{9}{2}.$$

(c) Write a parametrization of the reverse path.

Solution. $\gamma^r(t) = \gamma(1-t) = (1-t, 2(1-t))$.

(d) Find the integral of $\mathbf{F}$ along γ^r .

Solution.

$$\int_{\gamma^r} \mathbf{F} \cdot d\mathbf{s} = - \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = -\frac{9}{2}.$$

27. For each of the following, decide whether $\mathbf{F}$ is conservative on the indicated domain D . If it is, give a potential.

(a) $\mathbf{F}(x, y) = (x, x)$, $D = \mathbb{R}^2$.

Solution. No: $\partial F_2/\partial x - \partial F_1/\partial y = 1 \neq 0$.

(b) $\mathbf{F}(x, y) = (2xe^y + y, x^2e^y + x - 2y)$, $D = \mathbb{R}^2$.

Solution. Yes: $\phi(x, y) = x^2e^y + xy - y^2$.

28. Let $\mathbf{F}(x, y) = (-y, x)$. Evaluate, in terms of x_1, x_2, y_1, y_2 , the line integral of $\mathbf{F}$ along the segment from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$.

Solution. This is the same computation as above:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = x_1y_2 - x_2y_1.$$

Geometrically this is the signed parallelogram area $\overrightarrow{OP} \wedge \overrightarrow{OQ}$.

29. Evaluate $\int_C \nabla \phi \cdot d\mathbf{s}$, where $\phi(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$ and C is the segment from $(3, 4)$ to $(5, 12)$.

Solution. The field is conservative on $\mathbb{R}^2 \setminus \{(0, 0)\}$, and the segment does not meet the origin, so

$$\int_C \nabla \phi \cdot d\mathbf{s} = \phi(5, 12) - \phi(3, 4) = \frac{1}{13} - \frac{1}{5} = -\frac{8}{65}.$$