

UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

EXERCISES ON UNCONSTRAINED OPTIMIZATION

Prof. Philip Pennance

1. Let $f(x, y) = \ln(x^2 + y^2 + 1)$. Find the critical point(s) of f and use the Hessian to determine whether they are maxima or minima.

Solution.

$$\begin{aligned} D_1 f(x, y) &= \frac{2x}{x^2 + y^2 + 1}, & D_2 f(x, y) &= \frac{2y}{x^2 + y^2 + 1}, \\ D_1^2 f(x, y) &= \frac{2(y^2 - x^2 + 1)}{(x^2 + y^2 + 1)^2}, & D_2^2 f(x, y) &= \frac{2(x^2 - y^2 + 1)}{(x^2 + y^2 + 1)^2}, \\ D_1 D_2 f(x, y) &= \frac{-4xy}{(x^2 + y^2 + 1)^2} = D_2 D_1 f(x, y). \end{aligned}$$

The gradient vanishes if and only if $(x, y) = (0, 0)$. At $P = (0, 0)$,

$$\mathcal{H}(P) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \det \mathcal{H}(P) = 4 > 0, \quad D_1^2 f(P) = 2 > 0.$$

Thus $\mathcal{H}(P)$ is positive definite and P is a strict local minimum. (In fact $f \geq f(0, 0) = 0$, so the minimum is global.)

2. Let $f(x, y) = 1 + x + 2y^2 + x^3$. Compute the second-degree Taylor approximation

$$f(p + h) = f(p) + (h \cdot \nabla) f(p) + \frac{1}{2!} (h \cdot \nabla)^2 f(p) + R_2$$

at the point $p = (2, 1)$.

Solution.

$$\begin{aligned} D_1 f(x, y) &= 1 + 3x^2, & D_2 f(x, y) &= 4y, & D_1^2 f(x, y) &= 6x, \\ D_2^2 f(x, y) &= 4, & D_1 D_2 f(x, y) &= 0, & D_2 D_1 f(x, y) &= 0. \end{aligned}$$

Evaluating at $p = (2, 1)$ gives $f(p) = 13$ and

$$\begin{aligned} D_1 f(p) &= 13, & D_2 f(p) &= 4, & D_1^2 f(p) &= 12, \\ D_2^2 f(p) &= 4, & D_1 D_2 f(p) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} f(p + h) &= f(p) + h_1 D_1 f(p) + h_2 D_2 f(p) \\ &\quad + \frac{1}{2} [h_1^2 D_1^2 f(p) + 2h_1 h_2 D_1 D_2 f(p) + h_2^2 D_2^2 f(p)] + R_2 \\ &= 13 + 13h_1 + 4h_2 + 6h_1^2 + 2h_2^2 + R_2, \end{aligned}$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{R_2}{\|h\|^2} = 0.$$

3. Compute the second-degree Taylor expansion at $p = (0, 0)$ of $f(x, y) = e^x \cos y$.

Solution.

$$\begin{aligned} D_1 f(x, y) &= e^x \cos y, & D_1 f(p) &= 1, \\ D_2 f(x, y) &= -e^x \sin y, & D_2 f(p) &= 0, \\ D_1^2 f(x, y) &= e^x \cos y, & D_1^2 f(p) &= 1, \\ D_2^2 f(x, y) &= -e^x \cos y, & D_2^2 f(p) &= -1, \\ D_1 D_2 f(x, y) &= -e^x \sin y, & D_1 D_2 f(p) &= 0. \end{aligned}$$

Hence

$$\begin{aligned} f(p + h) &= f(p) + h_1 D_1 f(p) + h_2 D_2 f(p) \\ &\quad + \frac{1}{2} [h_1^2 D_1^2 f(p) + 2h_1 h_2 D_1 D_2 f(p) + h_2^2 D_2^2 f(p)] + O(\|h\|^3) \\ &= 1 + h_1 + \frac{1}{2}(h_1^2 - h_2^2) + O(\|h\|^3). \end{aligned}$$

Writing $(h_1, h_2) = (x, y)$,

$$f(x, y) = 1 + x + \frac{1}{2}(x^2 - y^2) + \dots$$

4. Find and classify, by means of the Hessian, the critical point(s) of

$$f(x, y) = 2x^2 + y^2 - xy - 7y.$$

Solution.

$$\begin{aligned} D_1 f(x, y) &= 4x - y, & D_2 f(x, y) &= -x + 2y - 7, \\ D_1^2 f(x, y) &= 4, & D_2^2 f(x, y) &= 2, & D_1 D_2 f(x, y) &= -1. \end{aligned}$$

The equation $\nabla f(x, y) = (0, 0)$ is the linear system

$$4x - y = 0, \quad -x + 2y = 7,$$

with unique solution $(x, y) = (1, 4)$. The Hessian is constant:

$$\mathcal{H} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det \mathcal{H} = 7 > 0, \quad D_1^2 f = 4 > 0.$$

Thus $\mathcal{H}$ is positive definite and $(1, 4)$ is a strict local minimum.

5. Find the absolute maxima and minima of $f(x, y) = x^2 - xy + y^2$ on the square $S = [-1, 1] \times [-1, 1]$.

Solution. Absolute extrema occur at critical points in the interior or on the boundary.

Critical points.

$$\nabla f(x, y) = (2x - y, -x + 2y) = (0, 0) \iff (x, y) = (0, 0).$$

There $f(0, 0) = 0$.

Boundary. Parametrize the four sides by

$$\begin{aligned}\gamma_1(t) &= (1, t), & \gamma_2(t) &= (-1, t), & -1 \leq t \leq 1, \\ \gamma_3(t) &= (t, 1), & \gamma_4(t) &= (t, -1), & -1 \leq t \leq 1.\end{aligned}$$

Then

$$\begin{aligned}f \circ \gamma_1(t) &= f \circ \gamma_3(t) = 1 - t + t^2 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4}, \\ f \circ \gamma_2(t) &= f \circ \gamma_4(t) = 1 + t + t^2 = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4}.\end{aligned}$$

On $[-1, 1]$ each of these quadratics has minimum value $3/4$, and the endpoint values are 1 and 3. The value 3 occurs at

$$\gamma_1(-1) = (1, -1) \quad \text{and} \quad \gamma_2(1) = (-1, 1).$$

Comparing interior and boundary values, the global minimum on S is 0 at $(0, 0)$, and the global maximum on S is 3 at $(1, -1)$ and $(-1, 1)$.

6. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ be points in $\mathbb{R}^2$. Find the extreme points of

$$f(X) = \|X - P_1\|^2 + \|X - P_2\|^2 + \|X - P_3\|^2,$$

where $X = (x, y)$.

Solution. Expanding the definition,

$$f(x, y) = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2.$$

Hence

$$\nabla f(x, y) = 2(3x - x_1 - x_2 - x_3, 3y - y_1 - y_2 - y_3),$$

and

$$\nabla f(x, y) = \vec{0} \iff (x, y) = \frac{P_1 + P_2 + P_3}{3},$$

the centroid of the three points. The second derivatives are constant: $D_1^2 f = D_2^2 f = 6$ and $D_1 D_2 f = 0$, so

$$\mathcal{H} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \det \mathcal{H} = 36 > 0, \quad D_1^2 f = 6 > 0.$$

The critical point is therefore a strict local minimum.

Remark. The same conclusion holds, without calculus, for n points $\mathbf{X}_1, \dots, \mathbf{X}_n$ in $\mathbb{R}^m$. Let

$$\mathbf{C} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \quad \text{and} \quad b = \frac{1}{n} \sum_{i=1}^n \|\mathbf{X}_i\|^2$$

be the centroid and the mean squared norm, and set $f(\mathbf{X}) = \sum_{i=1}^n \|\mathbf{X} - \mathbf{X}_i\|^2$. Then

$$\begin{aligned} f(\mathbf{X}) &= \sum_{i=1}^n (\mathbf{X} - \mathbf{X}_i) \cdot (\mathbf{X} - \mathbf{X}_i) \\ &= n\|\mathbf{X}\|^2 - 2n\mathbf{X} \cdot \mathbf{C} + nb \\ &= n\|\mathbf{X} - \mathbf{C}\|^2 + n(b - \|\mathbf{C}\|^2). \end{aligned}$$

The second term is constant, so f attains its global minimum at the centroid $\mathbf{C}$, and the minimum value is $n(b - \|\mathbf{C}\|^2)$.