

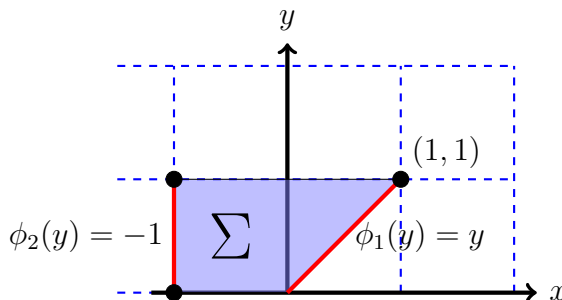
UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

PRACTICE PROBLEMS FOR EXAM III

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1. Evaluate $\iint_{\Sigma} f(x, y) dx dy$ where $f(x, y) = xy - 2y$ and $\Sigma \subseteq \mathbb{R}^2$ is the region given by $0 \leq y \leq 1$ and $-1 \leq x \leq y$.

Solution



$$\begin{aligned} \int_{\Sigma} f &= \int_0^1 \int_{x=-1}^{x=y} (xy - 2y) dx dy \\ &= \int_0^1 \left[\frac{yx^2}{2} - 2yx \right]_{-1}^y dy \\ &= \int_0^1 \left(\left[\frac{y^3}{2} - 2y^2 \right] - \left[\frac{y}{2} + 2y \right] \right) dy \\ &= \int_0^1 \left(\frac{y^3}{2} - 2y^2 - \frac{5y}{2} \right) dy \\ &= \left[\frac{y^4}{8} - \frac{2y^3}{3} - \frac{5y^2}{4} \right]_0^1 \\ &= -\frac{43}{24}. \end{aligned}$$

2. Evaluate the double integral $\int_1^2 \int_3^5 (3y - 4x) dx dy$ using the definition, in terms of lower and upper sums.

Solution

Let

$$\begin{aligned} P_1 &= \{3 = x_0 < x_1 < \cdots < x_m = 5\} \\ P_2 &= \{1 = y_0 < y_1 < \cdots < y_n = 2\} \end{aligned}$$

be partitions of $[3, 5]$ and $[1, 2]$ respectively, and let $P = P_1 \times P_2$. For each (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$ let $\pi_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ be the $(i, j)^{\text{th}}$ subrectangle of

the partition P . On the subrectangle π_{ij} the maximum M_{ij} of f occurs at (x_{i-1}, y_j) and the minimum m_{ij} at (x_i, y_{j-1}) , since f is decreasing in x and increasing in y .

Hence,

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^m \sum_{j=1}^n (3y_{j-1} - 4x_i) \Delta x_i \Delta y_j \\
&= 3 \sum_{j=1}^n y_{j-1} \Delta y_j \sum_{i=1}^m \Delta x_i - 4 \sum_{i=1}^m x_i \Delta x_i \sum_{j=1}^n \Delta y_j \\
&= \frac{3}{2} \sum_{j=1}^n (y_{j-1} + y_j) \Delta y_j \sum_{i=1}^m \Delta x_i - 2 \sum_{i=1}^m (x_i + x_{i-1}) \Delta x_i \sum_{j=1}^n \Delta y_j \\
&\leq \frac{3}{2} \sum_{j=1}^n (y_j + y_{j-1}) \Delta y_j \sum_{i=1}^m \Delta x_i - 2 \sum_{i=1}^m (x_i + x_{i-1}) \Delta x_i \sum_{j=1}^n \Delta y_j \\
&= \frac{3}{2} \sum_{j=1}^n (y_j^2 - y_{j-1}^2) \sum_{i=1}^m \Delta x_i - 2 \sum_{i=1}^m (x_i^2 - x_{i-1}^2) \sum_{j=1}^n \Delta y_j.
\end{aligned}$$

The sums are telescoping, so

$$\begin{aligned}
L(f, P) &\leq \frac{3}{2} (y_n^2 - y_0^2) (x_m - x_0) - 2 (x_m^2 - x_0^2) (y_n - y_0) \\
&= \frac{3}{2} (2^2 - 1^2) (5 - 3) - 2 (5^2 - 3^2) (2 - 1) \\
&= -23.
\end{aligned}$$

A similar argument shows that

$$\begin{aligned}
U(f, P) &\geq \frac{3}{2} (y_n^2 - y_0^2) (x_m - x_0) - 2 (x_m^2 - x_0^2) (y_n - y_0) \\
&= -23,
\end{aligned}$$

and therefore

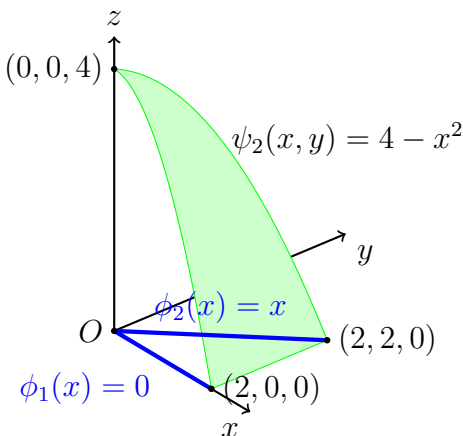
$$L(f, P) \leq -23 \leq U(f, P).$$

Since f is continuous on a compact rectangle it is integrable, and the integral is the unique number lying between every lower sum and every upper sum. As P was arbitrary,

$$\int_1^2 \int_3^5 (3y - 4x) dx dy = -23.$$

3. Evaluate $\int_{\Omega} f$ where $f(x, y, z) = xyz$ and $\Omega \subseteq \mathbb{R}^3$ is the region between the surfaces ψ_1 and ψ_2 defined for $0 \leq x \leq 2$ and $0 \leq y \leq x$ by $\psi_2(x, y) = 4 - x^2$ and $\psi_1(x, y) = 0$.

Solution



Let $\psi_2(x, y) = 4 - x^2$, $\psi_1(x, y) = 0$, $\phi_2(x) = x$, and $\phi_1(x) = 0$.

$$\begin{aligned} \int_{\Omega} f &= \int_0^2 \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x, y)}^{\psi_2(x, y)} xyz \, dz \, dy \, dx \\ &= \int_0^2 \int_0^x xy \frac{(4 - x^2)^2}{2} \, dy \, dx \\ &= \int_0^2 \frac{x^3}{4} (4 - x^2)^2 \, dx \\ &= \int_0^2 \left(\frac{x^7}{4} - 2x^5 + 4x^3 \right) \, dx \\ &= \left[\frac{x^8}{32} - \frac{x^6}{3} + x^4 \right]_0^2 \\ &= 8 - \frac{64}{3} + 16 = \frac{8}{3}. \end{aligned}$$

4. Let $X(r, \theta) = (r \cos \theta, r \sin \theta)$ be the usual polar coordinate map.

- (a) Find the matrix of the linear map $DX(r, \theta)$ and the corresponding Jacobian determinant at a point (r, θ) .

Solution

$$DX(r, \theta) = \begin{pmatrix} \frac{\partial X_1}{\partial r} & \frac{\partial X_1}{\partial \theta} \\ \frac{\partial X_2}{\partial r} & \frac{\partial X_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}, \quad \det DX(r, \theta) = r.$$

- (b) Use polar coordinates to find the integral of

$$f(x, y) = \sqrt{R^2 - (x^2 + y^2)}$$

over the disk of radius R centered at the origin.

Solution

Disregarding a set of measure zero, the image $X[\Omega]$ of the set $\Omega = [0, R] \times [0, 2\pi)$ in the (r, θ) plane is the disk of radius R centered at the origin. [See Java Applet](#). The change of variable formula

$$\iint_{X(\Omega)} f(x, y) dx dy = \iint_{\Omega} f(r \cos \theta, r \sin \theta) |r| dr d\theta$$

yields, since $r \geq 0$,

$$\begin{aligned} \iint_{X(\Omega)} \sqrt{R^2 - (x^2 + y^2)} dx dy &= \int_0^{2\pi} \int_0^R \sqrt{R^2 - r^2} \cdot r dr d\theta \\ &= 2\pi \int_0^R r \sqrt{R^2 - r^2} dr. \end{aligned}$$

The substitution $u = R^2 - r^2$, $du = -2r dr$ gives

$$\int_0^R r \sqrt{R^2 - r^2} dr = \frac{1}{2} \int_0^{R^2} u^{1/2} du = \frac{1}{3} R^3,$$

and therefore

$$\iint_{X(\Omega)} \sqrt{R^2 - (x^2 + y^2)} dx dy = \frac{2\pi R^3}{3}.$$

5. Sketch the region Ω in the first octant bounded above by the parabolic surface $z = 9 - x^2$ below by the plane $z = 3 - x$ and on the sides by the planes $y = 0$ and $y = 3$, and calculate its volume $V(\Omega)$.

Solution

The surfaces $z = 9 - x^2$ and $z = 3 - x$ meet when $x^2 - x - 6 = 0$, so $x = 3$ or $x = -2$. In the first octant they intersect along the segment $x = 3$, $z = 0$, $0 \leq y \leq 3$. On $0 \leq x \leq 3$ one has $9 - x^2 \geq 3 - x$. Thus

$$\begin{aligned} V(\Omega) &= \int_0^3 \int_0^3 \int_{3-x}^{9-x^2} dz dy dx \\ &= \int_0^3 \int_0^3 (9 - x^2 - (3 - x)) dy dx \\ &= 3 \int_0^3 (6 - x^2 + x) dx \\ &= 3 \left[6x - \frac{x^3}{3} + \frac{x^2}{2} \right]_0^3 \\ &= 3 \left(6 \cdot 3 - \frac{3^3}{3} + \frac{3^2}{2} \right) \\ &= 3 \left(18 - 9 + \frac{9}{2} \right) \\ &= \frac{81}{2}. \end{aligned}$$

6. Let $f(x, y) = x^2y$ and let C be the segment from $(3, 2)$ to $(7, 5)$. Compute the line integral $\int_C f(x, y) ds$.

Solution

$$\begin{aligned}\gamma(t) &= (3, 2) + t(4, 3), \quad 0 \leq t \leq 1. \\ x(t) &= 3 + 4t, \quad y(t) = 2 + 3t. \\ \gamma'(t) &= (4, 3), \quad \|\gamma'(t)\| = 5. \\ ds &= \|\gamma'(t)\| dt = 5 dt. \\ \int_C x^2y ds &= \int_0^1 (3 + 4t)^2(2 + 3t) \cdot 5 dt \\ &= \frac{3065}{6}.\end{aligned}$$

7. Let $f(x, y) = x^2y$ and let C be the segment from $(3, 2)$ to $(7, 5)$. Compute the integral $\int_C x^2y dx$.

Solution

$\int_C x^2y dx$ is the line integral of the vector field $F(x, y) = (x^2y, 0)$. Hence

$$\begin{aligned}\int_C x^2y dx &= \int_0^1 F(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_0^1 ((3 + 4t)^2(2 + 3t), 0) \cdot (4, 3) dt \\ &= \int_0^1 4(3 + 4t)^2(2 + 3t) dt \\ &= \frac{1226}{3}.\end{aligned}$$

8. Determine the work done by the force field $\mathbf{F}(x, y) = (x, y + 2)$ moving a point along the cycloid $\gamma(t) = (t - \sin t, 1 - \cos t)$, $0 \leq t \leq 2\pi$.

Solution

By inspection, $\mathbf{F}$ has a potential function $\phi(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + 2y$. Hence the work done is

$$\begin{aligned}W &= \phi(\gamma(2\pi)) - \phi(\gamma(0)) \\ &= \phi(2\pi, 0) - \phi(0, 0) \\ &= 2\pi^2.\end{aligned}$$

9. Let $\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ and let $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ be a path such that $\gamma(0) = (1, 1, 1)$ and $\gamma(1) = (2, 2, 2)$. Evaluate

(a) $\nabla\phi(x, y, z)$.

(b) $\int_{\gamma} \nabla \phi(x, y, z) \cdot d\mathbf{r}$

Solution

(a)

$$\nabla \phi(x, y, z) = -\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

(b) The function ϕ is C^∞ on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. Any path from $(1, 1, 1)$ to $(2, 2, 2)$ may be taken to miss the origin (for instance the straight segment), so

$$\int_{\gamma} \nabla \phi \cdot d\mathbf{r} = \phi(2, 2, 2) - \phi(1, 1, 1) = \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{6}.$$

10. Let $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$.

(a) Find the integral of the vector field F around the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Solution

Parametrize the circle counterclockwise by $\gamma(t) = (2 \cos t, 2 \sin t)$, $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_0^{2\pi} \left(\frac{-\sin t}{2}, \frac{\cos t}{2} \right) \cdot (-2 \sin t, 2 \cos t) dt \\ &= \int_0^{2\pi} 1 dt = 2\pi. \end{aligned}$$

(b) Let F_1 and F_2 be the components of F .

i. Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Solution

For $(x, y) \neq (0, 0)$,

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_1}{\partial y}.$$

Hence

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

ii. Are F_1 and F_2 continuous on $\mathbb{R}^2$?

No. They are not defined at the origin. In particular the domain $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is not simply connected.

iii. Is F conservative? Explain your answer.

F is not conservative on its domain: by part (a) the integral of F around a closed curve is nonzero.

11. Discuss briefly the geometric motivation underlying the definition of the integral of a scalar function over a smooth parametrized surface.

Solution

Let S be given by a smooth parametrization $X : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$. A small rectangle in the parameter domain with sides Δu and Δv is sent to a small parallelogram in the tangent plane spanned by $X_u \Delta u$ and $X_v \Delta v$, whose area is

$$\|X_u \times X_v\| \Delta u \Delta v.$$

If f is a scalar function on S (for instance a density), the contribution of that patch is approximately $f(X(u, v)) \|X_u \times X_v\| \Delta u \Delta v$. Summing and passing to the limit produces

$$\int_S f d\sigma = \iint_D f(X(u, v)) \|X_u \times X_v\| du dv.$$

When $f \equiv 1$ this is surface area; when f is a mass density it is the mass of S .

12. Show that the surface area of a sphere S of radius ρ is $4\pi\rho^2$ by calculating the surface integral $\int_S 1 d\sigma$.

Solution

Parametrize S by

$$X(\varphi, \theta) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi), \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$\begin{aligned} X_\varphi &= (\rho \cos \varphi \cos \theta, \rho \cos \varphi \sin \theta, -\rho \sin \varphi), \\ X_\theta &= (-\rho \sin \varphi \sin \theta, \rho \sin \varphi \cos \theta, 0), \end{aligned}$$

and

$$X_\varphi \times X_\theta = \rho^2 \sin \varphi (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi), \quad \|X_\varphi \times X_\theta\| = \rho^2 \sin \varphi.$$

Hence

$$\int_S 1 d\sigma = \int_0^{2\pi} \int_0^\pi \rho^2 \sin \varphi d\varphi d\theta = 2\pi\rho^2 [-\cos \varphi]_0^\pi = 2\pi\rho^2(1 + 1) = 4\pi\rho^2.$$

13. Evaluate the surface integral $\int_S yz d\sigma$, where S is the part of the plane $z = y + 3$ within the cylinder $x^2 + y^2 = 1$. (Answer: $\pi\sqrt{2}/4$.)

Solution

$$\begin{aligned} X(u, v) &= (u, v, v + 3), \quad u^2 + v^2 \leq 1, \\ X_u &= (1, 0, 0), \\ X_v &= (0, 1, 1), \\ X_u \times X_v &= (0, -1, 1), \\ \|X_u \times X_v\| &= \sqrt{2}. \end{aligned}$$

$$\begin{aligned}
\int_S yz \, d\sigma &= \sqrt{2} \iint_{u^2+v^2 \leq 1} v(v+3) \, du \, dv \\
&= \sqrt{2} \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (v^2 + 3v) \, dv \, du \\
&= \sqrt{2} \int_{-1}^1 \frac{2}{3} (1-u^2)^{3/2} \, du.
\end{aligned}$$

The substitution $u = \sin \theta$ gives

$$\int_{-1}^1 (1-u^2)^{3/2} \, du = 2 \int_0^{\pi/2} \cos^4 \theta \, d\theta = 2 \cdot \frac{3\pi}{16} = \frac{3\pi}{8},$$

and therefore

$$\int_S yz \, d\sigma = \sqrt{2} \cdot \frac{2}{3} \cdot \frac{3\pi}{8} = \frac{\pi\sqrt{2}}{4}.$$

14. Calculate the mass of the triangle with vertices $(1, 0, 0)$, $(1, 1, 1)$ and $(0, 0, 2)$ if the surface density is given by $\rho(x, y, z) = xz$.

Solution

The plane through the three vertices has equation $z = 2 - 2x + y$. The projection onto the xy -plane is the triangle with vertices $(1, 0)$, $(1, 1)$ and $(0, 0)$, described by $0 \leq u \leq 1$ and $0 \leq v \leq u$. A parametrization is

$$\begin{aligned}
X(u, v) &= (u, v, 2 - 2u + v), & 0 \leq u \leq 1, & \quad 0 \leq v \leq u, \\
X_u &= (1, 0, -2), \\
X_v &= (0, 1, 1), \\
X_u \times X_v &= (2, -1, 1), \\
\|X_u \times X_v\| &= \sqrt{6}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_S xz \, d\sigma &= \sqrt{6} \int_0^1 \int_0^u u(2 - 2u + v) \, dv \, du \\
&= \sqrt{6} \int_0^1 \left(2u^2 - \frac{3}{2}u^3 \right) \, du \\
&= \sqrt{6} \left[\frac{2}{3}u^3 - \frac{3}{8}u^4 \right]_0^1 = \frac{7\sqrt{6}}{24}.
\end{aligned}$$