

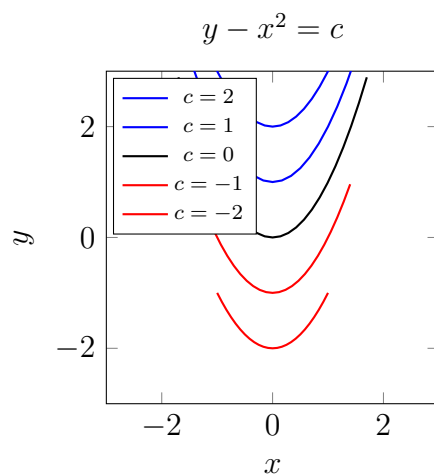
UNIVERSITY OF PUERTO RICO – RÍO PIEDRAS
DEPARTMENT OF MATHEMATICS

EXERCISES ON THE DERIVATIVE

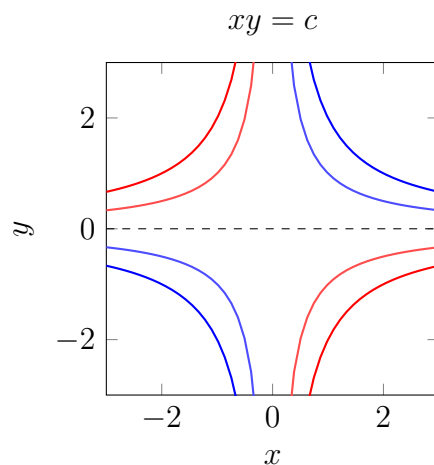
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1. Sketch at least five level curves of each of the following functions.

(a) $f(x, y) = y - x^2$



(b) $f(x, y) = xy$



2. Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ is linear if and only if there is a constant m such that $f(x) = mx$. Find and prove the analogous result for $f: \mathbb{R}^2 \rightarrow \mathbb{R}$.

Solution. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be linear. Then

$$f(x) = f(x \cdot 1) = x f(1).$$

Hence $f(x) = mx$ with $m = f(1)$. The converse is immediate.

Now let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be linear, and let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis of $\mathbb{R}^2$. If $x = (x_1, x_2) \in \mathbb{R}^2$, then

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2) \\ &= x_1 f(e_1) + x_2 f(e_2). \end{aligned}$$

Thus $f(x_1, x_2) = ax_1 + bx_2$ for constants $a = f(e_1)$ and $b = f(e_2)$. Conversely, every map of this form is linear.

Remark. Linear maps are determined by their values on a basis. In the one-dimensional case, $\{1\}$ is a basis of $\mathbb{R}$.

3. Let $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $L(x, y) = (L_1(x, y), L_2(x, y))$. If L is linear, show that the component functions L_1 and L_2 are linear.

Solution. Let $\pi_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the coordinate projections,

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

Then, for every $p \in \mathbb{R}^2$,

$$L_1(p) = \pi_1 \circ L(p), \quad L_2(p) = \pi_2 \circ L(p).$$

A composition of linear maps is linear, so each L_i is linear. Conversely, if both components are linear, then L is linear.

4. Find the partial derivatives D_1 , D_2 , and D_3 of $f(x, y, z) = x^{yz}$.

Solution.

$$\begin{aligned} D_1(x^{yz}) &= yz x^{yz-1}, \\ D_2(x^{yz}) &= D_2(e^{yz \ln x}) = z \ln x \cdot e^{yz \ln x} = z x^{yz} \ln x, \\ D_3(x^{yz}) &= y x^{yz} \ln x. \end{aligned}$$

5. Evaluate $\frac{\partial f}{\partial x}(0, 0)$ if $f(x, y) = \sqrt{xy}$.

Solution. The slice $y = 0$ is identically zero, so

$$D_1 f(0, 0) = [x \mapsto f(x, 0)]'(0) = [x \mapsto 0]'(0) = 0.$$

6. Let

$$f(x, y) = \frac{y \ln(x + \sqrt{x^2 - 1})}{x^2 + y^2}.$$

Evaluate $\frac{\partial f}{\partial y}(1, 4)$.

Solution. The numerator vanishes at $x = 1$, since $\ln(1 + \sqrt{1 - 1}) = \ln 1 = 0$. Thus the slice $x = 1$ is identically zero, and

$$D_2 f(1, 4) = [y \mapsto f(1, y)]'(4) = [y \mapsto 0]'(4) = 0.$$

7. Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $h(x, t) = e^{-t} \cos(x/c)$. Show that h satisfies the heat equation

$$\frac{\partial h}{\partial t} = c^2 \frac{\partial^2 h}{\partial x^2}.$$

Solution.

$$\frac{\partial h}{\partial t} = -e^{-t} \cos(x/c), \quad \frac{\partial h}{\partial x} = -\frac{1}{c} e^{-t} \sin(x/c), \quad \frac{\partial^2 h}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos(x/c).$$

Therefore

$$c^2 \frac{\partial^2 h}{\partial x^2} = -e^{-t} \cos(x/c) = \frac{\partial h}{\partial t}.$$

8. Let $\vec{n} \in \mathbb{R}^3$ be a unit vector. The amplitude of a wave at time t at a point with position vector $\vec{x} = (x_1, x_2, x_3)$ is

$$f(t, \vec{x}) = \sin(\vec{n} \cdot \vec{x} - ct).$$

Show that

$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \right).$$

Solution. Write $f(t, \vec{x}) = \sin(n_1 x_1 + n_2 x_2 + n_3 x_3 - ct)$. Then

$$\begin{aligned} \frac{\partial f}{\partial t} &= -c \cos(\vec{n} \cdot \vec{x} - ct), & \frac{\partial^2 f}{\partial t^2} &= -c^2 \sin(\vec{n} \cdot \vec{x} - ct), \\ \frac{\partial f}{\partial x_i} &= n_i \cos(\vec{n} \cdot \vec{x} - ct), & \frac{\partial^2 f}{\partial x_i^2} &= -n_i^2 \sin(\vec{n} \cdot \vec{x} - ct). \end{aligned}$$

Since $\|\vec{n}\|^2 = n_1^2 + n_2^2 + n_3^2 = 1$,

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = -\|\vec{n}\|^2 \sin(\vec{n} \cdot \vec{x} - ct) = -\sin(\vec{n} \cdot \vec{x} - ct) = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}.$$

9. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by $x \mapsto \|x\|$. Find the directional derivatives $D_{e_i} f(x)$ for $1 \leq i \leq 3$.

Solution.

$$\begin{aligned} f(x_1, x_2, x_3) &= \sqrt{x_1^2 + x_2^2 + x_3^2}, \\ \nabla f(x) &= \frac{x}{\|x\|}. \end{aligned}$$

Hence, for $x \neq 0$,

$$D_{e_i} f(x) = \nabla f(x) \cdot e_i = \frac{x_i}{\|x\|}, \quad 1 \leq i \leq 3.$$

10. Let $f(x, y) = x + y^3$, let $p = (1, 3)$, and let $\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

(a) Find $\nabla f(p)$.

(b) Find the directional derivative $D_{\vec{u}} f(p)$.

- (c) Compute $D_{\vec{u}}f(p)$ directly from the definition.
- (d) In which direction is the directional derivative maximal? What is its value in that direction? In a perpendicular direction?

Solution.

$$\nabla f(x, y) = (1, 3y^2),$$

$$\nabla f(1, 3) = (1, 27),$$

$$D_{\vec{u}}f(p) = \nabla f(p) \cdot \vec{u} = (1, 27) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 14\sqrt{2}.$$

From the definition,

$$f(p + t\vec{u}) = 1 + t\frac{\sqrt{2}}{2} + \left(3 + t\frac{\sqrt{2}}{2}\right)^3,$$

$$\frac{f(p + t\vec{u}) - f(p)}{t} = \frac{\sqrt{2}}{2} + 3 \cdot 9 \cdot \frac{\sqrt{2}}{2} + O(t).$$

Letting $t \rightarrow 0$ again yields $14\sqrt{2}$.

The directional derivative is maximal in the direction of $\nabla f(p)$, that is, in the direction of $\vec{v} = e_1 + 27e_2$. The maximal value is

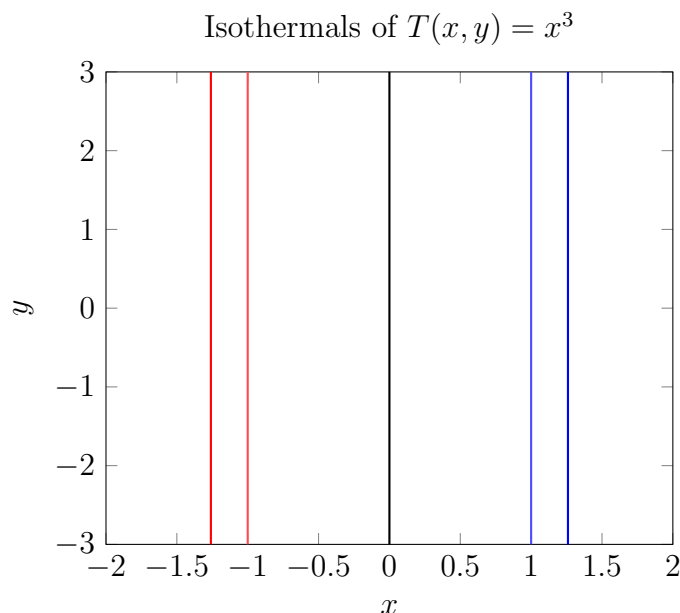
$$\|\nabla f(1, 3)\| = \sqrt{1 + 27^2} = \sqrt{730}.$$

In any direction perpendicular to $\nabla f(p)$ the directional derivative is 0.

11. The temperature at a point (x, y) in the plane is given by $T(x, y) = x^3$.

- (a) Sketch the isothermals of T for $T \in \{-2, -1, 0, 1, 2\}$.
- (b) Evaluate the derivative DT at the point $(3, 4)$.
- (c) Find the rate of increase of temperature at $(3, 4)$ in the direction $(1, 1)$.
- (d) Find the equation of the normal line to the surface $z = T(x, y)$ at $(1, 1)$.

Solution. The level set $x^3 = c$ is the vertical line $x = c^{1/3}$.



Since T does not depend on y ,

$$DT(3,4)(h,k) = 3x^2|_{(3,4)}h = 27h.$$

The unit vector in the direction $(1,1)$ is $\vec{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, so the rate of increase is

$$DT(3,4)(\vec{u}) = \frac{27\sqrt{2}}{2}.$$

The graph $z = x^3$ may be written $F(x, y, z) = x^3 - z = 0$. Then $\nabla F = (3x^2, 0, -1)$ and $\nabla F(1, 1, 1) = (3, 0, -1)$. A parametric equation of the normal line at $(1, 1, 1)$ is

$$(x, y, z) = (1, 1, 1) + t(3, 0, -1).$$

12. The temperature at (x, y) is

$$f(x, y) = 10 + 6 \cos x \cos y + 3 \cos 2x + 4 \cos 3y.$$

Find the direction of greatest decrease of temperature at $(\pi/3, \pi/3)$.

Solution.

$$\begin{aligned} \nabla f(x, y) &= (-6 \sin x \cos y - 6 \sin 2x, -6 \cos x \sin y - 12 \sin 3y), \\ -\nabla f(\pi/3, \pi/3) &= \left(6 \sin \frac{\pi}{3} \cos \frac{\pi}{3} + 6 \sin \frac{2\pi}{3}, 6 \cos \frac{\pi}{3} \sin \frac{\pi}{3} + 12 \sin \pi \right) = \left(\frac{9\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right). \end{aligned}$$

Thus a direction of greatest decrease is $\vec{v} = (9, 1)$.

13. Let $\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$. Evaluate $\nabla \phi(x, y, z)$.

Solution. Let $X = (x, y, z)$ and $\|X\| = \sqrt{x^2 + y^2 + z^2}$. Then

$$\nabla \phi(X) = -\left(\frac{x}{\|X\|^3}, \frac{y}{\|X\|^3}, \frac{z}{\|X\|^3} \right) = -\frac{X}{\|X\|^3}.$$

14. If $X = (x, y, z)$, use the chain rule and the fact that $\nabla \|X\| = X/\|X\|$ for $X \neq 0$ to show that

$$\nabla \frac{1}{\|X\|} = -\frac{X}{\|X\|^3}.$$

Solution. Let $r(\xi) = 1/\xi$. Then

$$\nabla \left(\frac{1}{\|X\|} \right) = \nabla (r \circ \|X\|) = r'(\|X\|) \nabla \|X\| = -\frac{1}{\|X\|^2} \cdot \frac{X}{\|X\|} = -\frac{X}{\|X\|^3}.$$

Physicists will recognize this as the field of an inverse-square attractive central force; up to sign, ϕ is the corresponding potential.

15. Find the coefficients a and b such that $Df(x, y)(h, k) = ah + bk$, where

$$f(x, y) = \int_0^{xy^2} e^{t^2} dt.$$

Solution. By the fundamental theorem and the chain rule,

$$a = D_1 f(x, y) = e^{(xy^2)^2} y^2, \quad b = D_2 f(x, y) = e^{(xy^2)^2} \cdot 2xy.$$

16. Find the derivative $Df(p)(\vec{h})$ if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f(x, y) = 2x + \sin y$.

Solution.

$$Df(p)(h_1, h_2) = \langle \nabla f(p), \vec{h} \rangle = \langle (2, \cos y), (h_1, h_2) \rangle = 2h_1 + (\cos y) h_2.$$

17. Find $Df(p)(\vec{h})$ if $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by $f(x, y) = (2x + 3y, 4x + 5y)$.

Solution. Since f is linear, $Df(p) = f$ at every point p . Hence

$$Df(x, y)(h_1, h_2) = f(h_1, h_2) = (2h_1 + 3h_2, 4h_1 + 5h_2).$$

18. Find the linear map $Df(x, y)$ if

$$f(x, y) = \left(2x + 3y, \int_1^{x^2+y} e^{-t^2} dt \right).$$

Also find the Jacobian matrix of f .

Solution. Write $f = (M, N)$ with $M(x, y) = 2x + 3y$ and $N(x, y) = \int_1^{x^2+y} e^{-t^2} dt$. Then $D_1 M = 2$ and $D_2 M = 3$. If F is an antiderivative of e^{-t^2} , then $N(x, y) = F(x^2 + y) - F(1)$, so

$$\begin{aligned} D_1 N(x, y) &= 2x e^{-(x^2+y)^2}, \\ D_2 N(x, y) &= e^{-(x^2+y)^2}. \end{aligned}$$

The Jacobian is

$$Jf(x, y) = \begin{pmatrix} 2 & 3 \\ 2xe^{-(x^2+y)^2} & e^{-(x^2+y)^2} \end{pmatrix},$$

and

$$Df(x, y)(h, k) = (2h + 3k, 2xe^{-(x^2+y)^2}h + e^{-(x^2+y)^2}k).$$

19. Suppose

$$f(x, y) = \int_{x^2y}^{x+y^2} e^{-t^2} dt.$$

Find a and b such that $Df(1, 2)(\vec{h}) = ah_1 + bh_2$.

Solution. If $F' = e^{-t^2}$, then $f(x, y) = F(x + y^2) - F(x^2y)$, and

$$\begin{aligned} D_1 f(x, y) &= e^{-(x+y^2)^2} - 2xy e^{-(x^2y)^2}, \\ D_2 f(x, y) &= 2y e^{-(x+y^2)^2} - x^2 e^{-(x^2y)^2}. \end{aligned}$$

Evaluating at $(1, 2)$ gives

$$a = e^{-25} - 4e^{-4}, \quad b = 4e^{-25} - e^{-4}.$$

20. Let $p = (x, y, z)$ and $\vec{h} = (h_1, h_2, h_3)$. Find $Df(p)(\vec{h})$ if

$$f(x, y, z) = \int_{xy^2+z}^{yz} \sinh t \, dt.$$

Solution. This is the special case of the next problem with $g(t) = \sinh t$, $k(x, y, z) = yz$, and $l(x, y, z) = xy^2 + z$. Directly,

$$\begin{aligned} Df(p)(h_1, h_2, h_3) &= [-y^2 \sinh(xy^2 + z)]h_1 + [z \sinh(yz) - 2xy \sinh(xy^2 + z)]h_2 \\ &\quad + [y \sinh(yz) - \sinh(xy^2 + z)]h_3. \end{aligned}$$

21. Let f have the form

$$f(p) = \int_{l(p)}^{k(p)} g(t) \, dt,$$

where $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $l, k: \mathbb{R}^3 \rightarrow \mathbb{R}$ are differentiable. Show that

$$Df(p)(\vec{h}) = g(k(p)) \nabla k(p) \cdot \vec{h} - g(l(p)) \nabla l(p) \cdot \vec{h}.$$

Solution. Let G be an antiderivative of g . By the fundamental theorem, $f = G \circ k - G \circ l$. The chain rule gives

$$D(G \circ k)(p)(\vec{h}) = DG(k(p))(Dk(p)(\vec{h})).$$

Since $G: \mathbb{R} \rightarrow \mathbb{R}$, one has $DG(t)(\tau) = g(t)\tau$, and $Dk(p)(\vec{h}) = \nabla k(p) \cdot \vec{h}$. Therefore

$$D(G \circ k)(p)(\vec{h}) = g(k(p)) \nabla k(p) \cdot \vec{h}.$$

The corresponding identity for l yields the claim.

Alternative argument. Let $\psi(p) = (l(p), k(p))$ and $\phi(\ell, \kappa) = \int_{\ell}^{\kappa} g$. Then $f = \phi \circ \psi$; apply the chain rule.

22. Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be bilinear, and let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Using the definition of the derivative as a linear map, show that

$$Df(\mathbf{a}, \mathbf{b})(\mathbf{x}, \mathbf{y}) = f(\mathbf{a}, \mathbf{y}) + f(\mathbf{x}, \mathbf{b}).$$

Solution. Bilinearity implies

$$f(\mathbf{a} + \mathbf{x}, \mathbf{b} + \mathbf{y}) = f(\mathbf{a}, \mathbf{b}) + f(\mathbf{a}, \mathbf{y}) + f(\mathbf{x}, \mathbf{b}) + f(\mathbf{x}, \mathbf{y}).$$

The remainder $f(\mathbf{x}, \mathbf{y})$ is $o(\|(\mathbf{x}, \mathbf{y})\|)$ as $(\mathbf{x}, \mathbf{y}) \rightarrow 0$, because $|f(\mathbf{x}, \mathbf{y})| \leq C\|\mathbf{x}\|\|\mathbf{y}\|$. Hence the candidate derivative $L(\mathbf{x}, \mathbf{y}) = f(\mathbf{a}, \mathbf{y}) + f(\mathbf{x}, \mathbf{b})$ satisfies the definition of $Df(\mathbf{a}, \mathbf{b})$.

23. Let $f(x, y) = 3x + 2y$. Find

- (a) the linear map $Df(1, 1)$;
- (b) the directional derivative of f at $(1, 1)$ in the direction $(3, 4)$;
- (c) a normal vector to the plane $3x + 2y - z = 0$ at $(1, 1, 5)$.

Solution. Since f is linear, $Df(a, b) = f$ at every point, so $Df(1, 1)(h, k) = 3h + 2k$.

If $\vec{u} = (\frac{3}{5}, \frac{4}{5})$ and $p = (1, 1)$, then

$$D_{\vec{u}}f(p) = Df(p)(\vec{u}) = f(\vec{u}) = \frac{17}{5}.$$

Let $F(x, y, z) = 3x + 2y - z$. Then $\nabla F = (3, 2, -1)$ is normal to the plane.

24. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x, y) = (2xy, x^2 + 3y^2)$. Find a function ϕ such that $f = \nabla\phi$.

Solution. $D_1\phi = 2xy$ implies $\phi(x, y) = x^2y + \psi(y)$. Then

$$D_2\phi = x^2 + \psi'(y) = x^2 + 3y^2,$$

so $\psi'(y) = 3y^2$ and $\psi(y) = y^3 + C$. Thus any function of the form

$$\phi(x, y) = x^2y + y^3 + C$$

works.

25. (a) Find the constant a such that $D \sin(\pi/3)(h) = ah$.

(b) Let $f(x, y) = 2x + 3x^2y^3$. Find a and b such that $Df(3, 4)(h, k) = ah + bk$.

Solution. $a = \cos(\pi/3) = 1/2$.

$$a = D_1f(3, 4) = [x \mapsto 2x + 3 \cdot 4^3x^2]'(3) = 2 + 3 \cdot 64 \cdot 2 \cdot 3 = 1154,$$

$$b = D_2f(3, 4) = 3x^2 \cdot 3y^2|_{(3,4)} = 9 \cdot 9 \cdot 16 = 1296.$$

26. (a) Let $f(x) = 2x$. Find $Df(3)(2)$.

(b) Let $f(x, y) = (2x + 3y, 4x + 5y)$ and $\alpha, \beta \in \mathbb{R}$. Find $Df(1, 1)(\alpha, \beta)$.

Solution. $Df(3)(2) = f'(3) \cdot 2 = 4$.

Since f is linear, $Df(1, 1) = f$, so $Df(1, 1)(\alpha, \beta) = (2\alpha + 3\beta, 4\alpha + 5\beta)$.

27. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by $f(x, y, z) = x^2yz$.

(a) Find the Jacobian matrix of $Df(x, y, z)$.

(b) Find the linear map $Df(1, 1, 1)$.

(c) Find the directional derivative of f at $(1, 1, 1)$ in the direction $(1, 2, 3)$.

Solution. The Jacobian (row) matrix is

$$Jf(x, y, z) = (2xyz, x^2z, x^2y).$$

Hence

$$Df(x, y, z)(h, k, l) = 2xyz h + x^2z k + x^2y l,$$

and $Df(1, 1, 1)(h, k, l) = 2h + k + l$.

A unit vector in the direction $(1, 2, 3)$ is $\vec{u} = (1, 2, 3)/\sqrt{14}$, so

$$D_{\vec{u}}f(1, 1, 1) = \frac{2 + 2 + 3}{\sqrt{14}} = \frac{7}{\sqrt{14}} = \sqrt{\frac{7}{2}}.$$

28. Find an equation of the plane tangent to the surface

$$\sin(xy) + \sin(yz) + \sin(xz) = 1$$

at $p = (1, \pi/2, 0)$.

Solution. Let $F(x, y, z) = \sin(xy) + \sin(yz) + \sin(xz)$. Then

$$\nabla F = (y \cos(xy) + z \cos(xz), x \cos(xy) + z \cos(yz), y \cos(yz) + x \cos(xz)),$$

and $\nabla F(1, \pi/2, 0) = (0, 0, \pi/2 + 1)$. The tangent plane is

$$(0, 0, \frac{\pi}{2} + 1) \cdot (x - 1, y - \frac{\pi}{2}, z) = 0,$$

which reduces to $z = 0$.

29. Let $f(x, y) = x^3 + 2y^2$.

(a) Find a and b such that $Df(1, 2)(\vec{h}) = ah_1 + bh_2$.

(b) Find the rate of increase of f in the direction $\vec{v} = (3, 4)$ at $p = (1, 1)$.

(c) Find a normal vector to the surface $z = f(x, y)$ at $(1, 2, 9)$.

(d) Find a normal vector to the level curve of f at $(2, 5)$.

Solution. $\nabla f(x, y) = (3x^2, 4y)$, so $\nabla f(1, 2) = (3, 8)$ and $a = 3$, $b = 8$.

At $(1, 1)$, $\nabla f = (3, 4)$ and a unit vector in the direction of $\vec{v}$ is $(\frac{3}{5}, \frac{4}{5})$, hence

$$D_{\vec{v}}f(1, 1) = (3, 4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = 5.$$

Writing the graph as $F(x, y, z) = x^3 + 2y^2 - z = 0$, one has $\nabla F(1, 2, 9) = (3, 8, -1)$.

A normal to the level curve at $(2, 5)$ is $\nabla f(2, 5) = (12, 20)$.