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DEPARTMENT OF MATHEMATICS

PRACTISE PROBLEMS ON PARAMETRIZED CURVES

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1. Let α be the curve given by $\alpha(t) = (2 \cos 3t, 2 \sin 3t)$, $0 \leq t \leq 2\pi$. Find:

- (a) A unit speed curve β with the same image as α .

Solution.

$$\alpha'(t) = (-6 \sin 3t, 6 \cos 3t).$$

Let $c \in (0, 2\pi)$ and consider the arclength from c

$$\begin{aligned} s &= \int_c^t \|\alpha'(\tau)\| d\tau \\ &= \int_c^t 6 d\tau \\ &= 6(t - c). \end{aligned}$$

Since adding a single value ($t = 0$) to the domain has no effect on arclength we can assume $c = 0$. Reparametrizing by the inverse function $t = \frac{s}{6}$, $0 \leq s < 12\pi$ yields a unit speed curve

$$\begin{aligned} \hat{\alpha}(s) &= \alpha\left(\frac{s}{6}\right) \\ &= \left(2 \cos \frac{s}{2}, 2 \sin \frac{s}{2}\right). \end{aligned}$$

- (b) The curvature κ of β .

Solution.

$$\begin{aligned} \kappa(s) &= \|\hat{\alpha}''(s)\| \\ &= \left\| \left(-\frac{1}{2} \cos \frac{s}{2}, -\frac{1}{2} \sin \frac{s}{2}\right) \right\| \\ &= \frac{1}{2}. \end{aligned}$$

2. (a) Find a parametrization $\gamma(t)$ of the upper half of the unit circle with the property that $\gamma(0) = (1, 0)$ and $\gamma(1) = (-1, 0)$.

Solution. The usual parametrization $\alpha(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$, traces the upper half-circle but is too slow to reach $(-1, 0)$ at $t = 1$. A constant speed equal to the arclength π works, so take $\gamma(t) = (\cos \pi t, \sin \pi t)$.

- (b) Find a parametrization $\alpha(t)$, $0 \leq t \leq 1$, of the upper half of the unit circle with $\alpha(0) = (-1, 0)$ and $\alpha(1) = (1, 0)$.

Solution. As t goes from 0 to 1, the term $1 - t$ goes from 1 to 0. Hence

$$\alpha(t) = (\cos \pi(1 - t), \sin \pi(1 - t)), \quad 0 \leq t \leq 1$$

reverses the curve of part (a).

- (c) Find a unit speed reparametrization of the curve α in part (b).

Solution. The parametrization in (b) has constant speed π , so $s = \pi t$ for $0 \leq t \leq 1$. Reparametrizing by arclength gives

$$\begin{aligned}\hat{\alpha}(s) &= \alpha(s/\pi) \\ &= (\cos(\pi - s), \sin(\pi - s)), \quad 0 \leq s \leq \pi.\end{aligned}$$

Remark: $\hat{\alpha}$ is the reverse of $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$.

3. (a) Write the line $2x - 4y + 5 = 0$ in the vector form $P + [u]$ where $P \in \mathbb{R}^2$ and $\vec{u}$ is a unit vector.

Solution. Let $P = (0, 5/4)$. Since $(2, -4)$ is normal to the line, $\vec{v} = (4, 2)$ is a direction vector. Take

$$u = \frac{v}{\|v\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right).$$

- (b) Let $P = (4, 3)$ and $\vec{u} = (1, 2)$. Write the image of $\gamma(t) = P + tu$, $t \in \mathbb{R}$, in the form $y = mx + b$.

Solution.

$$\gamma(t) = (4, 3) + t(1, 2).$$

$$\begin{aligned}x &= 4 + t, \\ y &= 3 + 2t.\end{aligned}$$

Eliminating t gives $y = 3 + 2(x - 4) = 2x - 5$.

4. Find an equation of the line:

- (a) normal to the curve $\gamma(t) = (\cos 3t, \sin 3t)$ at $t = \pi/3$.

Solution.

$$\begin{aligned}\gamma(\pi/3) &= (-1, 0), \\ \gamma'(t) &= (-3 \sin 3t, 3 \cos 3t), \\ \gamma'(\pi/3) &= (0, -3).\end{aligned}$$

A unit tangent is $T = (0, -1)$, so a normal direction is $N = (1, 0)$. The normal line is

$$X(t) = (-1, 0) + t(1, 0).$$

- (b) incident to the point $P(3, 4, 5)$ and parallel to the line $\gamma(t) = (1 + t, 2 - t, 5t)$.

Solution.

$$\gamma(t) = (1, 2, 0) + t(1, -1, 5).$$

A parallel line through $(3, 4, 5)$ is

$$X(t) = (3, 4, 5) + t(1, -1, 5).$$

5. Find an equation of the plane π normal to the curve $\gamma(t) = (e^t, t, t^2)$ at $t = 1$.

Solution.

$$\gamma(1) = (e, 1, 1), \quad \gamma'(t) = (e^t, 1, 2t), \quad \gamma'(1) = (e, 1, 2).$$

The plane through $p = (e, 1, 1)$ with normal $n = (e, 1, 2)$ satisfies

$$\begin{aligned} n \cdot X &= n \cdot p \\ ex + y + 2z &= e^2 + 3. \end{aligned}$$

6. Let $\gamma(t) = (t - \sin t, 1 - \cos t)$, $0 \leq t \leq 2\pi$. Find a unit speed reparametrization of γ starting at $t_0 = 0$. What is the domain of the reparametrized curve?

Solution.

$$\begin{aligned} \gamma'(t) &= (1 - \cos t, \sin t), \\ \|\gamma'(t)\| &= 2 \sin(t/2) \quad (0 \leq t \leq 2\pi). \end{aligned}$$

Arclength:

$$s = \int_0^t 2 \sin(\tau/2) d\tau = -4 \cos(t/2) + 4.$$

Inverting,

$$t = 2 \cos^{-1}((4 - s)/4), \quad 0 \leq s \leq 8.$$

The unit-speed curve is $\hat{\gamma}(s) = \gamma(t(s))$ on the domain $[0, 8]$.

7. Find the length of the curve $x = \ln \sec y$, $0 \leq y \leq \pi/3$.

Solution. It is equivalent to compute the length of $y = \ln \sec x$, $0 \leq x \leq \pi/3$, with $\gamma(x) = (x, \ln \sec x)$.

$$\begin{aligned} \|\gamma'(x)\| &= \sec x, \\ L &= \int_0^{\pi/3} \sec x dx = \ln(2 + \sqrt{3}). \end{aligned}$$

8. Find the length of each of the following curves:

(a) $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 3$.

Solution.

$$L = \int_1^3 \sqrt{1 + [f'(x)]^2} dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \frac{53}{6}.$$

(b) $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$.

Solution.

$$L = \int_{-r}^r \frac{r}{\sqrt{r^2 - x^2}} dx = \pi r.$$

9. Find the length of the curve given in polar coordinates by $r = e^\theta$, $0 \leq \theta \leq 2\pi$.

Solution.

$$\gamma(\theta) = (e^\theta \cos \theta, e^\theta \sin \theta), \quad \|\gamma'(\theta)\| = \sqrt{2} e^\theta.$$

Hence,

$$\begin{aligned} s &= \int_0^{2\pi} \|\gamma'(\tau)\| d\tau \\ &= \sqrt{2} e^\tau \Big|_0^{2\pi} \\ &= \sqrt{2} (e^{2\pi} - 1). \end{aligned}$$

10. Find the radius of curvature of the curve $y = x^2$ at the point $(2, 4)$.

Solution.

$$\kappa(x) = \frac{2}{(1 + 4x^2)^{3/2}}, \quad \kappa(2) = \frac{2}{17\sqrt{17}}, \quad R = \frac{17\sqrt{17}}{2}.$$

11. Find the curvature of the curve $y = \tan^{-1} x$ when $x = 1$.

Solution.

$$\begin{aligned} k(x) &= \frac{|y''(x)|}{[1 + y'(x)^2]^{\frac{3}{2}}} \\ &= \frac{\left| \frac{-2x}{1 + x^2} \right|}{\left[1 + \left(\frac{1}{1+x^2} \right)^2 \right]^{\frac{3}{2}}} \\ &= \frac{2x(1 + x^2)}{[(1 + x^2)^2 + 1]^{\frac{3}{2}}}. \end{aligned}$$

Hence, if $x = 1$

$$k(1) = \frac{4}{\sqrt{5}^3}; \quad R = \frac{1}{k} = \frac{\sqrt{5}^3}{4}.$$

12. Let $\gamma(t) = (t - \sin t, 1 - \cos t)$. Find the minimum value of the curvature on $0 \leq t \leq 2\pi$.
Note: $1 - \cos t = 2 \sin^2(t/2)$.

Solution.

$$\kappa(t) = \frac{1}{4|\sin(t/2)|}.$$

The minimum value $\kappa = 1/4$ occurs at $t = \pi$.

13. Find a differentiable curve whose image is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Where does the ellipse have maximum and minimum curvature?

Solution. Take $\gamma(t) = (a \cos t, b \sin t)$, $0 \leq t \leq 2\pi$. Then

$$\kappa(t) = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}.$$

If $a > b$, curvature is maximum at $t = 0, \pi$ and minimum at $t = \pi/2$.

14. Let $\gamma(t) = (r \cos t, r \sin t, ct)$ where $r > 0$ and c are constant. Find a unit speed reparametrization of γ .

Solution.

$$\|\gamma'(t)\| = \sqrt{r^2 + c^2} =: v.$$

With $t = s/v$,

$$\hat{\gamma}(s) = (r \cos(s/v), r \sin(s/v), cs/v).$$

15. Let P be a point and n a unit vector in $\mathbb{R}^3$. Let M, N be two distinct points in the plane π through P with normal n . Show that the line through M and N is contained in π .

Solution. $X \in \pi$ iff $n \cdot (X - P) = 0$. If $X = tM + (1 - t)N$, then $n \cdot (X - P) = 0$, so $X \in \pi$.

16. Let $A = (1, 2, 7)$ and $B = (4, 6, 7)$. Find a unit speed parametrization of the segment from A to B .

Solution. The direction vector of the segment is

$$B - A = (3, 4, 0).$$

Its length is

$$\|B - A\| = \sqrt{3^2 + 4^2} = 5,$$

so a unit direction vector is

$$u = \left(\frac{3}{5}, \frac{4}{5}, 0\right).$$

A unit-speed parametrization starting at A when $s = 0$ and ending at B when $s = 5$ is

$$\hat{\gamma}(s) = A + su = \left(1 + \frac{3s}{5}, 2 + \frac{4s}{5}, 7\right), \quad 0 \leq s \leq 5.$$

Equivalently, if $\gamma(t) = A + t(B - A)$ for $0 \leq t \leq 1$, then $\hat{\gamma}(s) = \gamma(s/5)$.

Check:

$$\hat{\gamma}'(s) = \left(\frac{3}{5}, \frac{4}{5}, 0\right), \quad \|\hat{\gamma}'(s)\| = 1,$$

$\hat{\gamma}(0) = A$, and $\hat{\gamma}(5) = B$.

17. (a) Parametrize the graph of $f(x) = \cosh x$, $0 \leq x \leq \log 2$.

Solution.

$$\gamma(t) = (t, \cosh t), \quad 0 \leq t \leq \ln 2.$$

- (b) Write the equation of the reverse of the parametrization in (a).

Solution.

$$\gamma^r(t) = (\ln 2 - t, \cosh(\ln 2 - t)), \quad 0 \leq t \leq \ln 2.$$

- (c) Find the arclength of the parametrized curve in (a).

Solution.

$$s = \int_0^{\ln 2} \cosh x \, dx = \sinh(\ln 2) = \frac{3}{4}.$$