

DEPARTMENT OF MATHEMATICS

MATE 3153 – EXAM 4 SEMESTER 2, 2022-23

Instructions: *Time allowed 2 hours. All computations and work must shown to obtain credit. Please start each question on a new side. questions will be*

Last Names: _____ First Name: _____
Student # : _____ Professor: Philip Pennance Section: _____

Answer SIX Questions Only.

1. Find a vector parallel to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$. Find an equation for the line of intersection of the two planes.

Solution.

The planes have normal vectors $n_1 = (2, -1, 1)$ and $n_2 = (3, 1, 1)$. The cross product $\vec{v} = n_1 \times n_2 = (-2, 1, 5)$ is a direction vector to the line of intersection of the two planes. Putting $x = 0$ in the equations of the planes and solving for the other coordinates yields a point $P = (0, \frac{1}{2}, \frac{3}{2})$. in the intersection of the planes. Hence, an equation for the line of intersection is

$$\begin{aligned} X(t) &= P + tv, \quad t \in \mathbb{R} \\ &= (0, \frac{1}{2}, \frac{3}{2}) + t(-2, 1, 5) \quad t \in \mathbb{R}. \end{aligned}$$

2. Let $\vec{\alpha} = (-1, 2, -3)$ and $\vec{\beta} = (2, 3, 6)$.

- (a) Find the projection of $\vec{\alpha}$ on $\vec{\beta}$.

Solution.

$$\begin{aligned} \vec{\alpha} \cdot \vec{\beta} &= (-1)(2) + (2)(3) + (-3)(6) = -14. \\ \|\vec{\beta}\| &= \sqrt{2^2 + 3^2 + 6^2} = 7. \\ P_{\vec{\beta}}(\vec{\alpha}) &= \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} \vec{\beta} \\ &= \frac{-14}{49} (2, 3, 6) \\ &= \left(-\frac{4}{7}, -\frac{6}{7}, -\frac{12}{7} \right) \end{aligned}$$

- (b) Find a parametrization $X(u, v)$ of the interior of the triangle with vertices $P = (1, 1, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 2)$. (Do not forget to state the domain of the parametrization.)

Solution.

$$X(u, v) = P + u(Q - P) + v(R - P) = (1 - u - v, 1 - v, 2v),$$

with domain

$$u \geq 0, \quad v \geq 0, \quad u + v \leq 1.$$

3. Find and test, by means of the Hessian, the critical point(s) of

$$f(x, y) = 2x^2 + y^2 - xy - 7y.$$

Solution.

$$\begin{array}{lll} D_1f(x, y) = 4x - y & D_2f(x, y) = -x + 2y - 7 & D_1^2f(x, y) = 4 \\ D_2^2f(x, y) = 2 & D_1D_2f(x, y) = -1 & D_2D_1f(x, y) = -1. \end{array}$$

Critical Points:

$$\begin{aligned} \nabla f(x, y) = (0, 0) &\iff (4x - y, -x + 2y - 7) = (0, 0) \\ &\iff (x, y) = (1, 4) \end{aligned}$$

Hessian:

$$\begin{aligned} \det \mathcal{H}(1, 4) &= \begin{vmatrix} D_1^2f(1, 4) & D_1D_2f(1, 4) \\ D_2D_1f(1, 4) & D_2^2f(1, 4) \end{vmatrix} \\ &= \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 7, \end{aligned}$$

Since $\det \mathcal{H}(1, 4) > 0$ and $D_1^2f(1, 4) > 0$, the Hessian is positive definite and the critical point is a local minimum.

4. Let $f(x, y) = x^2 + y$ and P the point $(1, 3)$. Compute

- (a) The gradient $\nabla f(P)$.
- (b) The linear map $Df(P)(h, k)$.
- (c) The rate of increase of f at the point P in the direction $\vec{u} = (1, 0)$.
- (d) A unit vector in the direction of the maximum rate of increase of f at P .

Solution.

$$\begin{aligned} \nabla f(x, y) &= (2x, 1) \\ \nabla f(1, 3) &= (2, 1) \end{aligned}$$

$$Df(P)(h, k) = \nabla f(P) \cdot (h, k) = (2, 1) \cdot (h, k) = 2h + k.$$

$$\begin{aligned} D_{\vec{u}}f(P) &= \nabla f(P) \cdot \vec{u} \\ &= (2, 1) \cdot (1, 0) \\ &= 2 \end{aligned}$$

A unit vector in the direction of maximum increase is

$$\frac{\nabla f(P)}{\|\nabla f(P)\|} = \frac{(2, 1)}{\sqrt{5}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right).$$

5. Let α be the curve given by $\alpha(t) = (2 \cos 3t, 2 \sin 3t)$, $0 < t < 2\pi$. find:

- (a) A unit speed curve β with the same image as α .
- (b) The curvature κ of β at any time t .

Solution.

$$\begin{aligned}\alpha'(t) &= (-6 \sin 3t, 6 \cos 3t) \\ s &= \int_0^t \|\alpha'(\tau)\| d\tau \\ &= \int_0^t 6 d\tau \\ &= 6t.\end{aligned}$$

Using the arclength as parameter yields a unit speed parametrization,

$$\beta(s) = (2 \cos(s/2), 2 \sin(s/2)).$$

The curvature $\kappa(s) = \|\beta''(s)\| = \frac{1}{2}$.

6. Let $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$.

- (a) Find the integral of the vector field F around the circle $x^2 + y^2 = 4$ oriented counter clockwise.

Solution.

Parametrize the circle counter clockwise by $\gamma(t) = (2 \cos t, 2 \sin t)$. Then

$$\begin{aligned}\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} (\mathbf{F}(\gamma(s)) \cdot \gamma'(s)) ds \\ &= \int_0^{2\pi} \left(\frac{-\sin s}{2}, \frac{\cos s}{2} \right) \cdot (-2 \sin s, 2 \cos s) ds \\ &= 2\pi.\end{aligned}$$

- (b) Let F_1 and F_2 be the components of F .

- i. Calculate $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Solution.

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_1}{\partial y}. \text{ Hence } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

- ii. Are F_1 and F_2 continuous on $\mathbb{R}^2$?

No. They do not exist at the origin. Remark: It follows that the domain is not simply connected.

iii. Is F conservative? Explain your answer.

Solution.

F is not conservative since, by part (a), the integral of F around a closed curve is non zero.

7. Use the method of Lagrange multipliers (and nothing but the method of Lagrange multipliers) to find the extreme points of $f(x, y) = x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$.

Solution. Let $g(x, y) = x^2 + y^2$.

$$\begin{aligned}\nabla f(x, y) &= \lambda \nabla g(x, y) \\ \iff (2x, -2) &= (2\lambda x, 2\lambda y)\end{aligned}$$

Hence,

$$(\lambda - 1)x = 0 \tag{1}$$

$$\lambda y = -1 \tag{2}$$

$$x^2 + y^2 = 1 \tag{3}$$

I. If $x = 0$, then $y = \pm 1$.

II. If $x \neq 0$ then $\lambda = 1$ and so $y = -1$. The constraint yields $x = 0$ – a contradiction.

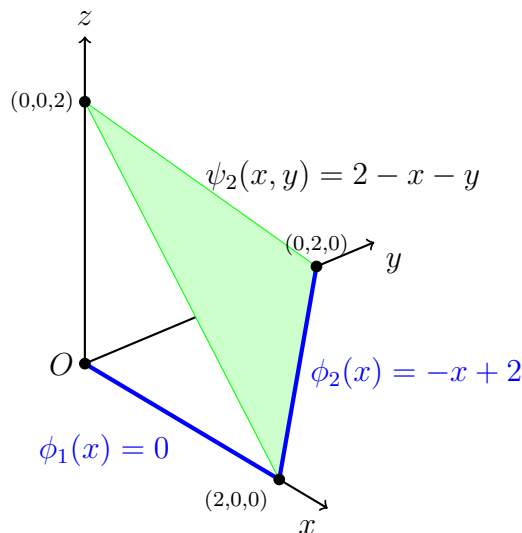
Hence $x = 0$ is the only possibility. Now, $f(0, 1) = -2$, $f(0, -1) = 2$

and so $(0, -1)$ is a maximum point and $(0, 1)$ a minimum.

8. Evaluate $\iiint_V 1 \, dx \, dy \, dz$ where V is the three dimensional region in the first octant bounded by the planes $x + y + z = 2$, $x = 0$, $y = 0$, and $z = 0$.

Solution.

The region has the form of a tetrahedron.



Let $\psi_2(x, y) = 2 - x - y$; $\psi_1(x, y) = 0$; $\phi_2(x) = -x + 2$; $\phi_1(x) = 0$.

$$\begin{aligned}\int_T 1 &= \int_0^2 \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} 1 \, dz \, dy \, dx \\ &= \int_0^2 \int_0^{-x+2} \int_0^{2-x-y} 1 \, dz \, dy \, dx \\ &= \int_0^2 \int_0^{-x+2} 2 - x - y \, dy \, dx \\ &= \frac{4}{3}.\end{aligned}$$

9. Let $X(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$.

(a) Compute the Jacobian determinant of X .

Solution.

$$J_X = \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \det J_X = r$$

(b) Evaluate the triple integral of the constant function $f(x, y, z) = 1$ over the region Ω described by the inequalities $0 \leq z \leq h$, and $x^2 + y^2 \leq \rho^2$ where $\rho, h > 0$ are constant.

Solution.

$$\begin{aligned}\int_{\Omega} 1 &= \int_0^{2\pi} \int_0^{\rho} \int_0^h r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\rho} rh \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \rho^2 h r \, d\theta \\ &= \pi \rho^2 h.\end{aligned}$$

10. Let $\mathbf{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$ and let S be the surface with parametrization

$$X(u, v) = (u, v, 1 - u - v), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u.$$

Compute the surface integral $\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

$$\begin{aligned}\mathbf{F}(x, y, z) &= (x + y^2, y + z^2, z + x^2) \\ \nabla \times \mathbf{F}(x, y, z) &= (-2z, -2x, -2y)\end{aligned}$$

$$X_u = (1, 0, -1)$$

$$X_v = (0, 1, -1)$$

$$X_u \times X_v = (1, 1, 1)$$

$$\nabla \times F(X(u, v)) = -2(1 - u - v, u, v).$$

$$\begin{aligned} \int_S \nabla \times F \cdot n \, d\sigma &= \int_0^1 \int_0^{1-u} \nabla \times F(X(u, v)) \cdot (X_u \times X_v) \, dv \, du. \\ &= \int_0^1 \int_0^{1-u} -2 \, dv \, du \\ &= -1. \end{aligned}$$