

DEPARTMENT OF MATHEMATICS

MATE 3153 – EXAM 3 SEMESTER 2, 2022-23

Instructions: *Time allowed 2 hours.* All computations and work must shown to obtain credit. Please start each question on a new side. questions will be

Last Names: _____ First Name: _____

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Answer FIVE Questions Only.

1. Use the definition, in terms of lower and upper sums, to evaluate the double integral

$$\int_3^5 \int_1^2 1 \, dx \, dy.$$

Solution. Let $R = [1, 2] \times [3, 5]$ and let $f \equiv 1$. Let

$$P = \{x_0, \dots, x_n\}, \quad Q = \{y_0, \dots, y_m\}$$

be arbitrary partitions of $[1, 2]$ and $[3, 5]$ respectively, and write

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

On every subrectangle one has

$$m_{ij} = \min_{R_{ij}} f = 1, \quad M_{ij} = \max_{R_{ij}} f = 1.$$

Consequently the lower and upper sums coincide with the area of R :

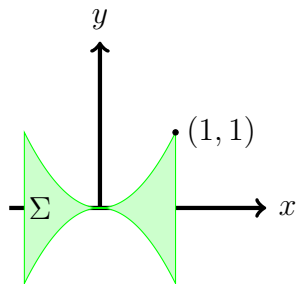
$$\begin{aligned} L(f, P, Q) &= \sum_{i=1}^n \sum_{j=1}^m m_{ij} \Delta x_i \Delta y_j = \sum_{i=1}^n \sum_{j=1}^m \Delta x_i \Delta y_j = 2, \\ U(f, P, Q) &= \sum_{i=1}^n \sum_{j=1}^m M_{ij} \Delta x_i \Delta y_j = 2. \end{aligned}$$

This holds for every partition P . Hence, f is integrable on R and

$$\int_3^5 \int_1^2 1 \, dx \, dy = 2.$$

2. Evaluate $\iint_{\Sigma} f \, dx \, dy$ where $f(x, y) = x^4 - y^3$ and $\Sigma \subseteq \mathbb{R}^2$ is the region given by $-1 \leq x \leq 1$; $-x^2 \leq y \leq x^2$.

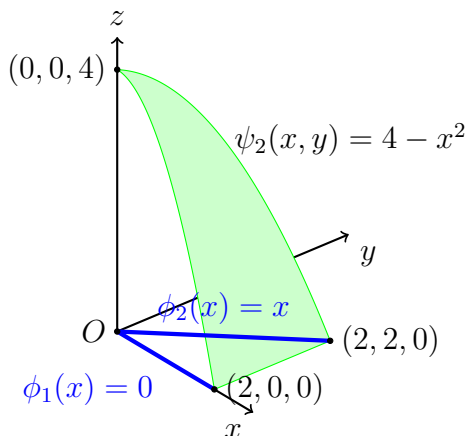
Solution.



$$\begin{aligned}
 \int_{\Sigma} f(x, y) \, dx \, dy &= \int_{-1}^1 \int_{-x^2}^{x^2} x^4 - y^3 \, dy \, dx \\
 &= \int_{-1}^1 \left[x^4 y - \frac{y^4}{4} \right]_{-x^2}^{x^2} dx \\
 &= \int_{-1}^1 2x^6 \, dx \\
 &= 4 \int_0^1 x^6 \, dx \\
 &= \frac{4}{7}
 \end{aligned}$$

3. Evaluate $\iiint_{\Omega} f(x, y, z) dx dy dz$ where $f(x, y, z) = xyz$ and $\Omega \subseteq \mathbb{R}^3$ is the region between the surfaces ψ_1 and ψ_2 defined for $0 \leq x \leq 2$ and $0 \leq y \leq x$ by $\psi_2(x, y) = 4 - x^2$ and $\psi_1(x, y) = 0$.

Solution.



Let $\psi_2(x, y) = 4 - x^2$; $\psi_1(x, y) = 0$; $\phi_2(x) = x$; $\phi_1(x) = 0$.

$$\begin{aligned} \int_{\Omega} f &= \int_0^2 \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x, y)}^{\psi_2(x, y)} xyz dz dy dx \\ &= \int_0^2 \int_0^x xy \frac{(4 - x^2)^2}{2} dy dx \\ &= \int_0^2 \frac{x^3}{4} (4 - x^2)^2 dx \\ &= \int_0^2 \frac{x^7}{4} - 2x^5 + 4x^3 dx \\ &= \frac{8}{3}.. \end{aligned}$$

4. Let $X(r, \theta) = (r \cos \theta, r \sin \theta)$ be the usual polar coordinate map.
- (a) Find the matrix of the linear map $DX(r, \theta)$ and the corresponding Jacobian determinant at a point (r, θ) .

Solution.

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial r} & \frac{\partial X_1}{\partial \theta} \\ \frac{\partial X_2}{\partial r} & \frac{\partial X_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}; \quad \det J = r.$$

- (b) Use polar coordinates to find the integral of $f(x, y) = \sqrt{R^2 - (x^2 + y^2)}$ over the disk of radius R centered at the origin.

Solution

Disregarding a set of measure zero, the image $X[\Omega]$ of the set $\Omega = [0, R] \times [0, 2\pi)$ in the (r, θ) plane is the disk of radius R centered at the origin. The change of variable formula

$$\int \int_{X(\Omega)} f(x, y) dx dy = \int \int_{\Omega} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

yields

$$\begin{aligned} \int \int_{X(\Omega)} \sqrt{R^2 - (x^2 + y^2)} dx dy &= \int_0^{2\pi} \int_0^R \sqrt{R^2 - r^2} \cdot r dr d\theta \\ &= \frac{2\pi R^3}{3}. \end{aligned}$$

5. Let $f(x, y) = x^2y$ and let C be the segment from (3,2) to (7,5). Compute the line integral $\int_C f(x, y) ds$.

Solution.

$$\gamma(t) = (3, 2) + t(4, 3), \quad 0 \leq t \leq 1.$$

$$x(t) = 3 + 4t; \quad y(t) = 2 + 3t.$$

$$\gamma'(t) = (4, 3); \quad \|\gamma'(t)\| = 5.$$

$$ds = \|\gamma'(t)\| dt$$

$$= 5 dt.$$

$$\begin{aligned} \int_C x^2y ds &= \int_0^1 (3 + 4t)^2(2 + 3t) \cdot 5 dt \\ &= \frac{3065}{6}. \end{aligned}$$

6. Let $f(x, y) = x^2y$ and let C be the segment from (3,2) to (7,5). Compute the integral $\int_C x^2y dx$.

Solution.

$\int_C x^2y dx$ is the line integral of the vector field $F(x, y) = (x^2y, 0)$. Hence

$$\begin{aligned} \int_C x^2y dx &= \int_0^1 F(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_0^1 ((3 + 4t)^2(2 + 3t), 0) \cdot (4, 3) dt \\ &= \int_0^1 4(3 + 4t)^2(2 + 3t) dt \\ &= \frac{1226}{3} \end{aligned}$$

7. Determine whether the force field $\mathbf{F}(x, y) = (x, y + 2)$ is conservative. Determine the work done by this force field while moving a point along the cycloid $\gamma(t) = (t - \sin t, 1 - \cos t)$, $0 \leq t \leq 2\pi$.

Solution.

By inspection, $\mathbf{F}$ has a potential function $\phi(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + 2y$. Hence the work done is

$$\begin{aligned} W &= \phi(\gamma(2\pi)) - \phi(\gamma(0)) \\ &= \phi(2\pi, 0) - \phi(0, 0) \\ &= 2\pi^2. \end{aligned}$$

8. A triangle of uniform mass density has vertices $A = (0, 0)$, $B = (b, 0)$ and $C = (0, h)$. Show, by integration, that the center of mass is located at the point $(b/3, h/3)$.

Solution. The triangle Σ may be described by

$$0 \leq x \leq b, \quad 0 \leq y \leq h\left(1 - \frac{x}{b}\right).$$

Its area is $bh/2$. With uniform density ρ the mass is therefore

$$M = \rho \cdot \frac{bh}{2}.$$

The coordinates of the centre of mass are

$$\bar{x} = \frac{1}{M} \iint_{\Sigma} x \rho dA = \frac{2}{bh} \int_0^b \int_0^{h(1-x/b)} x dy dx = \frac{2}{b} \int_0^b x \left(1 - \frac{x}{b}\right) dx = \frac{2}{b} \left[\frac{x^2}{2} - \frac{x^3}{3b}\right]_0^b = \frac{b}{3},$$

$$\bar{y} = \frac{1}{M} \iint_{\Sigma} y \rho dA = \frac{2}{bh} \int_0^b \int_0^{h(1-x/b)} y dy dx = \frac{1}{b} \int_0^b h \left(1 - \frac{x}{b}\right)^2 dx = \frac{h}{3}.$$

Hence the centre of mass is $(b/3, h/3)$.

9. Find a parametrization $X(u, v)$ of the hemisphere $z = \sqrt{9 - x^2 - y^2}$. Use the parametrization X to find an equation for the tangent plane at the point $(2, 2, 1)$.

Solution. One parametrization of the hemisphere is the graph map

$$X(u, v) = (u, v, \sqrt{9 - u^2 - v^2}), \quad u^2 + v^2 \leq 9.$$

Differentiating:

$$X_u = \left(1, 0, -\frac{u}{z}\right), \quad X_v = \left(0, 1, -\frac{v}{z}\right).$$

At the point $(2, 2, 1)$

$$X_u = (1, 0, -2), \quad X_v = (0, 1, -2).$$

A normal vector is

$$X_u \times X_v = (2, 2, 1).$$

The tangent plane is therefore

$$2(x - 2) + 2(y - 2) + (z - 1) = 0,$$

I.e.,

$$2x + 2y + z = 9.$$

10. Let $\mathbf{F}(x, y, z) = (x + y^2, y + z^2, z + x^2)$ and let

$$X(u, v) = (1 - u - v, u, v), \quad u \geq 0, v \geq 0, u + v \leq 1$$

be a parametrization of a surface S .

(a) Compute $\nabla \times \mathbf{F}$.

(b) Calculate the composition $(\nabla \times \mathbf{F})(X(u, v))$.

(c) Evaluate the surface integral $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$.

Solution.

(a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x + y^2 & y + z^2 & z + x^2 \end{vmatrix} = (-2z, -2x, -2y).$$

(b) Substituting the parametrization yields

$$(\nabla \times \mathbf{F})(X(u, v)) = (-2v, -2(1 - u - v), -2u).$$

(c) The partial derivatives of X are

$$X_u = (-1, 1, 0), \quad X_v = (-1, 0, 1),$$

so

$$X_u \times X_v = (1, 1, 1).$$

Hence

$$(\nabla \times \mathbf{F})(X(u, v)) \cdot (X_u \times X_v) = (-2v) - 2(1 - u - v) - 2u = -2.$$

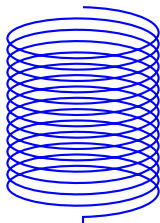
The parameter domain $D = \{(u, v) : u \geq 0, v \geq 0, u + v \leq 1\}$ has area $1/2$, and therefore

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_D (-2) \, du \, dv = -1.$$

11. The wire in a solenoid is non-uniform with mass density given by $\lambda(x, y, z) = z$. The form of the wire is the image of the curve

$$\gamma(t) = (2 \cos t, 2 \sin t, 3t); \quad 0 \leq t \leq 20\pi.$$

Find the total mass of the solenoid.



Solution.

Let M be the mass of the wire.

$$\begin{aligned} \gamma'(t) &= (-2 \sin t, 2 \cos t, 3) \\ \|\gamma'(t)\| &= \sqrt{13}. \\ M &= \int_{\gamma} \lambda \, ds \\ &= \int_{\gamma} \lambda(\gamma(t)) \cdot \|\gamma'(t)\| \, dt \\ &= \int_0^{20\pi} \lambda(2 \cos t, 2 \sin t, 3t) \cdot \sqrt{13} \, dt \\ &= \int_0^{20\pi} 3t \cdot \sqrt{13} \, dt \\ &= 3\sqrt{13} \int_0^{20\pi} t \, dt \\ &= 600\sqrt{13} \cdot \pi^2. \end{aligned}$$