

DEPARTMENT OF MATHEMATICS

MATE 3153 – EXAM 2 SEMESTER 2, 2022-23

Instructions: *Time allowed 2 hours.* All computations and work must shown to obtain credit. Please start each question on a new side. questions will be

Last Names: _____ First Name: _____

Student # : _____ Professor: Philip Pennance Section: _____

Answer FIVE Questions Only.

1. (a) Let $f(x, y) = \frac{y \ln(x + \sqrt{x^2 - 1})}{x^2 + y^2}$. Evaluate $\frac{\partial f}{\partial y}(1, 4)$.

Solution.

$$\begin{aligned} D_2 f(1, 4) &= [y \mapsto f(1, y)]'(4) \\ &= [y \mapsto 0]'(4) \\ &= 0. \end{aligned}$$

- (b) Let $f(x, y) = x^2 + y$. Let P be the point $(1, 3)$ and $\vec{u} = (1, 0)$.

Compute $D_{\vec{u}}f(P)$ directional derivative using only the definition.

Solution.

$$\begin{aligned} D_{\vec{u}}f(P) &= \lim_{t \rightarrow 0} \frac{f(P + t\vec{u}) - f(P)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(1 + t, 3) - f(1, 3)}{t} \\ &= \lim_{t \rightarrow 0} \frac{1 + 2t + t^2 + 3 - 4}{t} \\ &= \lim_{t \rightarrow 0} \frac{2t + t^2}{t} \\ &= 2. \end{aligned}$$

2. (a) Find the constant a such that $D \sin(\pi/3)(h) = ah$.

Solution.

$$a = \sin'(\pi/3) = \frac{1}{2}.$$

- (b) Let $f(x, y) = 2x + 3x^2y^3$. Find the constants a and b such that

$$Df(3, 4)(h, k) = ah + bk.$$

Solution.

$$\begin{aligned}a &= D_1 f(3, 4) \\&= [x \mapsto f(x, 4)]'(3) \\&= [x \mapsto 2x + 3 \cdot 4^3 x^2]'(3) \\&= [x \mapsto 2x + 3 \cdot 4^3 x^2]'(3) \\&= 2 + 3 \cdot 4^3 \cdot 2 \cdot 3 \\&= 1154.\end{aligned}$$

Similarly, $b = D_2 f(3, 4) = 1296$.

3. Find and test, by means of the Hessian, the critical point(s) of

$$f(x, y) = 2x^2 + y^2 - xy - 7y.$$

Solution.

$$\begin{array}{lll}D_1 f(x, y) = 4x - y & D_2 f(x, y) = -x + 2y - 7 & D_1^2 f(x, y) = 4 \\D_2^2 f(x, y) = 2 & D_1 D_2 f(x, y) = -1 & D_2 D_1 f(x, y) = -1.\end{array}$$

Critical Points:

$$\begin{aligned}\nabla f(x, y) = (0, 0) &\iff (4x - y, -x + 2y - 7) = (0, 0) \\&\iff (x, y) = (1, 4)\end{aligned}$$

Hessian:

$$\begin{aligned}\det \mathcal{H}(1, 4) &= \begin{vmatrix} D_1^2 f(1, 4) & D_1 D_2 f(1, 4) \\ D_2 D_1 f(1, 4) & D_2^2 f(1, 4) \end{vmatrix} \\&= \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} \\&= 7,\end{aligned}$$

Since $\det \mathcal{H}(1, 4) > 0$ and $D_1^2 f(1, 4) > 0$, the Hessian is positive definite and the critical point is a local minimum.

4. Let $f(x, y) = 1 + x + 2y^2 + x^3$. Compute the second degree Taylor approximation

$$f(p + h) = f(p) + (h \cdot \nabla) f(p) + \frac{1}{2!} (h \cdot \nabla)^2 f(p) + R_2.$$

at the point $p = (2, 1)$.

Solution.

$$\begin{array}{lll}D_1 f(x, y) = 1 + 3x^2 & D_2 f(x, y) = 4y & D_1^2 f(x, y) = 6x \\D_2^2 f(x, y) = 4 & D_1 D_2 f(x, y) = 0 & D_2 D_1 f(x, y) = 0.\end{array}$$

Therefore,

$$\begin{array}{lll} D_1 f(2, 1) = 13 & D_2 f(2, 1) = 4 & D_1^2 f(2, 1) = 12 \\ D_2^2 f(2, 1) = 4 & D_1 D_2 f(2, 1) = 0 & D_2 D_1 f(2, 1) = 0. \end{array}$$

$$\begin{aligned} f(p+h) &= f(p) + h_1 D_1 f(p) + h_2 D_2 f(p) \\ &\quad + \frac{1}{2} [h_1^2 D_1^2 f(p) + 2h_1 h_2 D_1 D_2 f(p) + h_2^2 D_2^2 f(p)] + R_2 \\ &= 13 + 13h_1 + 4h_2 + 6h_1^2 + 2h_2^2 + R_2. \end{aligned}$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{R_2}{\|h\|^2} = 0.$$

5. Find the maximum of the function

$$H(x, y, z) = -(x \ln x + y \ln y + z \ln z)$$

subject to the constraint $x + y + z = 1$.

Solution.

Let $g(x, y, z) = x + y + z$. Since g has no critical points, 1 is a regular value and the method of Lagrange applies. Hence, at a critical point, there exists λ such that $\nabla H = \lambda \nabla g$. It follows that

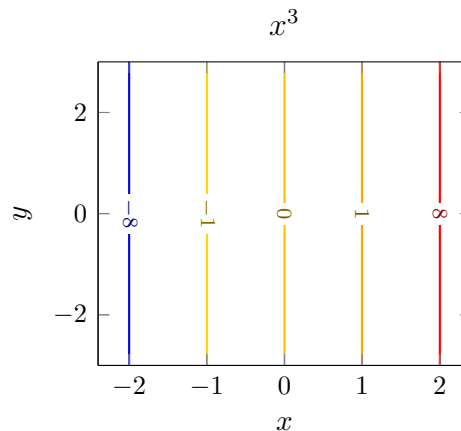
$$\begin{aligned} -\ln x - 1 &= \lambda \\ -\ln y - 1 &= \lambda \\ -\ln z - 1 &= \lambda \end{aligned}$$

It follows that $x = y = z$. Using the constraint equation, it must be that $x = y = z = \frac{1}{3}$. The maximum of H is

$$H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = -\ln \frac{1}{3} = \ln 3.$$

6. The temperature at a point (x, y) in the plane is given by $T(x, y) = x^3$.

(a) Sketch the isothermals (level curves) of f for $T \in \{-8, -1, 0, 1, 8\}$.



(b) Evaluate the derivative $DT(1, 1)(\vec{h})$ if $\vec{h} = (3, 4)$.

Solution.

$$DT(1, 1)(\vec{h}) = \nabla T(1, 1) \cdot \vec{h} = (3, 0) \cdot (3, 4) = 9.$$

(c) Find the rate of increase of temperature at $(1, 1)$ in the direction $\vec{h}$.

Solution.

Let $\vec{u} = (\frac{3}{5}, \frac{4}{5})$. The directional derivative is $\nabla T(1, 1) \cdot \vec{u} = 9/5$.

(d) Find the equation of the normal line to the surface $z = T(x, y)$ when $(x, y) = (1, 1)$.

Solution.

Let $g(x, y, z) = z - x^3$. Then $\nabla g = (-3x^2, 0, 1)$. The required line has direction vector $\nabla g(1, 1, 1) = (-3, 0, 1)$ and hence equation $X = (1, 1, 1) + t(-3, 0, 1), t \in \mathbb{R}$.

7. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x, y) = (2xy, x^2 + 3y^2)$. Find a function ϕ such that $f = \nabla \phi$.

Solution.

$$D_1 \phi(x, y) = 2xy \implies \phi(x, y) = x^2 y + \psi(y).$$

$$\begin{aligned} D_2 \phi(x, y) &= x^2 + \psi'(y) \\ &= x^2 + 3y^2 \end{aligned}$$

Hence $\psi'(y) = 3y^2$ and so $\psi(y) = y^3 + C$. Therefore any function of the form $\phi(x, y) = x^2 y + y^3 + C$ will do.