Math 3153 - Exam II. Practice Questions.

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1. Sketch the levels curves of height c

$$f(x,y) = \ln(x) - \ln(y)$$

for all

$$c \in \{-\ln(3), -\ln(2), 0, \ln(2), \ln(3)\}$$

2. Let

$$u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y.$$

Show that

$$D_1 u(x,y) = D_2 v(x,y)$$
$$D_2 u(x,y) = -D_1 v(x,y)$$

for all $(x,y) \in \mathbb{R}$

3. Let

$$u(x,y) = x^n - y^n.$$

Find all natural numbers n such that

$$u_{xx} + u_{yy} = 0.$$

4. Let

$$u(x,y) = e^x \cos y + e^y \cos x + xy$$

Find a function $v(x,y)$ such that

$$u_x = v_y$$
$$u_y = -v_x.$$

5. Let n be a Show that the function

$$\phi(x,t) = \sin(x - ct)$$

satisfies the one dimensional wave equation.

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}.$$

6. Let

$$f(x,y,z) = \ln zx + \cos xy + e^{yz}.$$

Find the directional derivative $D_u f(P)$ at the point $P = (\frac{1}{3}, 0, 3)$ in the direction $\vec{u} = (1,2,3)$.

7. Let $f(x,y) = x^2 + y$. Find an equation of the tangent plane to the graph of f at the point $(1,2,3)$.

8. Find $\dfrac{du}{dt}$ if

$$u(x,y) = x^2 - y^2$$
$$x(t) = \cos t$$
$$y(t) = \sin t.$$

9. Find the minimum and maximum values of

$$f(x,y) = 3x + 4y$$

subject to the constraint $x^2 + y^2 = 1$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = (2xy, x^2 + 3y^2).$$

Find a function ϕ such that $f = \nabla\phi$.

11. Let

$$F(x,y) = (x^2 y, y^3)$$

and

$$\gamma(t) = (t,t), \quad t \in [0,1].$$

(a) Show that the vector field F is not conservative.

(b) Calculate the integral of F along the path γ.

(c) Write the equation for the reverse of the path γ.

(d) Find the integral of F along the the reverse of the path γ.

[1] http://pennance.us

12. Let $\alpha(s) = \log_e(s)$, $s \in [1, e]$ and let $\hat{\gamma}(s) = \gamma(\alpha(s))$, $s \in [1, e]$. Calculate the integral of $F(x, y) = (x^2y, y^3)$ along the path $\hat{\gamma}$.

13. Find a degree 2 polynomial approximation of the function $f(x, y) = \log(1 + x + 2y)$ around $P = (2, 1)$.

14. Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field on an open set U and $\gamma : [a, b] \to U$ a continuously differentiable curve joining two points P, Q of U

    (a) If F is the gradient of a real valued differentiable function ϕ on U show that $\displaystyle\int_\gamma F$ is independent of γ.

15. Let $f(x, y) = \dfrac{y \ln\left(x + \sqrt{x^2 - 1}\right)}{x^2 + y^2}$.

    Evaluate $\dfrac{\partial f}{\partial y}(1, 4)$.

    Solution.

$$\begin{aligned} D_2 f(1, 4) &= [y \mapsto f(1, y)]'(4) \\ &= [y \mapsto 0]'(4) \\ &= 0. \end{aligned}$$

16. (a) Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$. Let (a, b) be a point in U. Define the partial derivative $\dfrac{\partial f}{\partial x}(a, b)$.

    (b) Evaluate $\dfrac{\partial f}{\partial y}(2, 3)$ if

$$f(x, y) = \frac{\sin(x - 2)}{1 + \sin(y^2 + \tan xy)}$$

    (c) Evaluate $\dfrac{\partial f}{\partial x}(0, 0)$ if

$$f(x, y) = \sqrt{xy}.$$

17. Let $n \in \mathbb{R}^3$ a unit vector. For each $t \in \mathbb{R}$ define a function $f_t : \mathbb{R}^3 \to \mathbb{R}$ by $f_t(x) = \sin(n \cdot x - 3t)$. Show that f_t is constant on any plane of the form $\langle n, x\rangle = d$. Give a physical interpretation.

18. Let

$$f(x, y) = \frac{x + y}{1 - xy}.$$

    Evaluate $D_2 f(2, 3)$.

19. Let $f(x, t) : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, t) = \sin(x - ct)$. Show that f satisfies the wave equation: $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$.