

DEPARTMENT OF MATHEMATICS

MATE 3153 – EXAM I SEMESTER 2, 2022-23

Instructions: *Time allowed 2 hours. All computations and work must shown to obtain credit. Please start each question on a new side. questions will be*

Last Names: _____ First Name: _____

Student # : _____ Professor: Philip Pennance Section: _____

Answer SIX Questions Only.

1. Show that the midpoints of the sides of a quadrilateral $ABCD$ in $\mathbb{R}^3$ form a parallelogram.

Solution.

The midpoints are:

$$M_1 = \frac{A + B}{2}$$

$$M_2 = \frac{B + C}{2}$$

$$M_3 = \frac{C + D}{2}$$

$$M_4 = \frac{D + A}{2}$$

Then,

$$\begin{aligned} M_1 - M_2 &= \frac{A - C}{2} \\ &= M_4 - M_3. \end{aligned}$$

$$\begin{aligned} M_1 - M_4 &= \frac{B - D}{2} \\ &= M_2 - M_3. \end{aligned}$$

It follows that M_1, M_2, M_3, M_4 are the vertices of a parallelogram.

2. Use vectors to show that the diagonals of a parallelogram coincide at the midpoints.

Solution.

Let $ABCD$ be a parallelogram with $\overrightarrow{AB} = \overrightarrow{DC}$. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the points A, B, C, D respectively. Then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{DC} \\ \iff \overrightarrow{OA} - \overrightarrow{OB} &= \overrightarrow{OD} - \overrightarrow{OC} \\ \iff \vec{a} - \vec{b} &= \vec{d} - \vec{c} \\ \iff \frac{\vec{a} + \vec{c}}{2} &= \frac{\vec{b} + \vec{d}}{2} \end{aligned}$$

It follows that the midpoints of diagonals AC and BD coincide.

3. (a) Write the line $2x - 4y + 5 = 0$ in the vector form $P + [u]$ where $P \in \mathbb{R}^2$ and $\vec{u}$ is a unit vector.

Solution.

Let $P = (0, \frac{4}{5})$. Since $(2, -4)$ is normal to the line, $\vec{v} = (4, 2)$ is a direction vector. Take $u = \frac{\vec{v}}{\|\vec{v}\|} = (\frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}}) = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$.

- (b) Let $P = (4, 3)$ and $\vec{u} = (1, 2)$, write the image of the curve $\gamma(t) = P + t\vec{u}$, $t \in \mathbb{R}$ in the form $y = mx + b$.

Solution.

$$\gamma(t) = (4, 3) + t(1, 2).$$

$$x = 4 + t$$

$$y = 3 + 2t$$

Eliminating t gives $y = 3 + 2(x - 4) = 2x - 5$.

4. (a) Find a parametrization $\gamma(t)$ of the upper half of the unit circle with the property that $\gamma(0) = (1, 0)$ and $\gamma(1) = (-1, 0)$

Solution.

Notice that the usual parametrization $\alpha(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$. Parametrizes the upper half of the unit circle but the speed is too slow to reach $(-1, 0)$ by $t = 1$. In order to reach $(-1, 0)$ after one unit of time a constant speed equal to the arclength π suffices. Hence, take $\gamma(t) = (\cos \pi t, \sin \pi t)$.

- (b) Find a parametrization $\alpha(t)$, $0 \leq t \leq 1$ of the upper half of the unit circle with the property that $\alpha(0) = (-1, 0)$ and $\alpha(1) = (1, 0)$

Solution.

Notice that as t increases from 0 to 1, the term $1 - t$ decreases from 1 to 0. Hence, the reparametrization

$$\alpha(t) = (\cos \pi(1 - t), \sin \pi(1 - t)), \quad 0 \leq t \leq 1$$

will reverse the curve obtained in (a).

- (c) Find a unit speed reparametrization of the curve α in part (b).

Solution.

The parametrization α in part (b) has constant speed π and so the arclength is given by

$$s = \pi t, \quad 0 \leq t \leq 1.$$

Reparametrization of α by arclength yields the unit speed curve:

$$\begin{aligned} \hat{\alpha}(s) &= \alpha(s/\pi) \\ &= (\cos(\pi - s), \sin(\pi - s)), \quad 0 \leq s \leq \pi. \end{aligned}$$

Remark: $\hat{\alpha}$ is just the reverse of $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$.

5. Find an equation of the plane π normal to the curve $\gamma(t) = (e^t, t, t^2)$ at the point $t = 1$.

Solution.

$$\begin{aligned}\gamma(1) &= (e, 1, 1) \\ \gamma'(t) &= (e^t, 1, 2t).\end{aligned}$$

Hence, the required plane π is incident to the point $p = (e, 1, 1)$ and has normal vector $n = \gamma'(1) = (e, 1, 2)$. Therefore π has equation

$$\begin{aligned}n \cdot X &= n \cdot p \\ (e, 1, 2) \cdot (x, y, z) &= (e, 1, 2) \cdot (e, 1, 1) \\ ex + y + 2z &= e^2 + 3.\end{aligned}$$

6. Find the length of the curve given in polar coordinates by $r = e^\theta$, $0 \leq \theta \leq 2\pi$.

Solution.

Converting to cartesian coordinates, the curve can be written

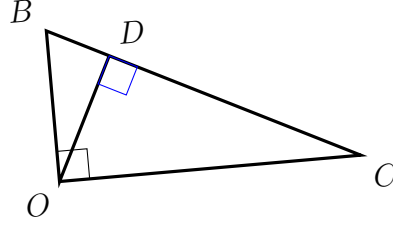
$$\begin{aligned}\gamma(\theta) &= (r \cos \theta, r \sin \theta) \\ &= (e^\theta \cos \theta, e^\theta \sin \theta) \\ \gamma'(\theta) &= e^\theta (\cos \theta - \sin \theta, \cos \theta + \sin \theta) \\ \|\gamma'\|^2 &= 2e^{2\theta} \\ \|\gamma'\| &= \sqrt{2}e^\theta\end{aligned}$$

Hence,

$$\begin{aligned}s &= \int_0^{2\pi} \|\gamma'(\tau)\| d\tau \\ &= \sqrt{2}e^\tau \Big|_0^{2\pi} \\ &= \sqrt{2}(e^{2\pi} - 1)\end{aligned}$$

7. Let $O = (0, 0, 0)$, B and C be the vertices of a right triangle in $\mathbb{R}^3$ with hypotenuse BC . Let D be the foot of the perpendicular from O to BC . Let $\vec{b}, \vec{c}, \vec{d}$ be the position vectors of B, C, D respectively. Find (in terms of B and C the constant λ such that

$$\vec{d} = (1 - \lambda)\vec{b} + \lambda\vec{c} \tag{1}$$



Solution.

Taking the dot product of (1) with $\vec{b}$ and with $\vec{c}$ and noting that $\vec{b} \cdot \vec{c} = 0$ yields

$$\vec{b} \cdot \vec{d} = (1 - \lambda)\|\vec{b}\|^2$$

$$\vec{c} \cdot \vec{d} = \lambda\|\vec{c}\|^2$$

Subtracting gives

$$(\vec{b} - \vec{c}) \cdot \vec{d} = (1 - \lambda)\|\vec{b}\|^2 - \lambda\|\vec{c}\|^2$$

Since $\vec{d}$ is perpendicular to $\vec{b} - \vec{c}$ the left hand side is zero. Hence

$$(1 - \lambda)\|\vec{b}\|^2 - \lambda\|\vec{c}\|^2 = 0$$

$$\implies \lambda = \frac{\|\vec{b}\|^2}{\|\vec{b}\|^2 + \|\vec{c}\|^2}$$

$$\implies 1 - \lambda = \frac{\|\vec{c}\|^2}{\|\vec{b}\|^2 + \|\vec{c}\|^2}$$

and so

$$\vec{d} = \frac{\|\vec{b}\|^2}{\|\vec{c}\|^2 + \|\vec{b}\|^2} \vec{b} + \frac{\|\vec{c}\|^2}{\|\vec{c}\|^2 + \|\vec{b}\|^2} \vec{c}.$$

8. Express $(3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta})$ as a scalar multiple of $\vec{\alpha} \times \vec{\beta}$.

Solution.

Since the cross product is bilinear and antisymmetric,

$$\begin{aligned} (3\vec{\alpha} + 2\vec{\beta}) \times (\vec{\alpha} + 4\vec{\beta}) &= 3(\vec{\alpha} \times \vec{\alpha}) + 12(\vec{\alpha} \times \vec{\beta}) + (2\vec{\beta} \times \vec{\alpha}) + 8(2\vec{\beta} \times \vec{\beta}) \\ &= 12(\vec{\alpha} \times \vec{\beta}) + 8(\vec{\beta} \times \vec{\alpha}) \\ &= 12(\vec{\alpha} \times \vec{\beta}) - 8(\vec{\alpha} \times \vec{\beta}) \\ &= 10(\vec{\alpha} \times \vec{\beta}). \end{aligned}$$

9. Let $\vec{\alpha} = (-1, 2, -3)$ and $\vec{\beta} = (2, 3, 6)$.

(a) Evaluate $\langle \vec{\alpha}, \vec{\beta} \rangle$.

Solution.

$$\langle \vec{\alpha}, \vec{\beta} \rangle = (-1)(2) + (2)(3) + (-3)(6) = -14.$$

(b) Evaluate $\|\vec{\beta}\|$.

Solution.

$$\|\vec{\beta}\| = \sqrt{2^2 + 3^2 + 6^2} = 7.$$

(c) Find the projection of $\vec{\alpha}$ on $\vec{\beta}$.

Solution.

$$\begin{aligned} P_{\vec{\beta}}(\vec{\alpha}) &= \frac{\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} \vec{\beta} \\ &= \frac{-14}{49} (2, 3, 6) \\ &= \left(-\frac{4}{7}, -\frac{6}{7}, -\frac{12}{7} \right) \end{aligned}$$

(d) Evaluate $\vec{\alpha} \times \vec{\beta}$.

Solution.

$$\begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix} = 21i + 0j - 7k = (21, 0, -7).$$

10. Find a differentiable curve whose image is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Where does the ellipse have maximum and minimum curvature?

Solution

Let

$$\gamma(t) = (x(t), y(t)), \quad 0 \leq t \leq 2\pi$$

where $x = a \cos t$ and $y = b \sin t$. Then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also,

$$\begin{aligned} \gamma'(t) &= (-a \sin t, b \cos t) \\ \gamma''(t) &= (-a \cos t, -b \sin t) \\ \|\gamma'(t)\| &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \end{aligned}$$

The curvature κ is given by

$$\begin{aligned}\kappa &= \frac{|\gamma' \wedge \gamma''|}{|\gamma'|^3} \\ &= \frac{|x'y'' - x''y'|}{\left[\sqrt{x'^2 + y'^2}\right]^3} \\ &= \frac{ab}{\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{3/2}}\end{aligned}$$

Now,

$$a^2 \sin^2 t + b^2 \cos^2 t = (a^2 - b^2) \sin^2 t + b^2.$$

If $a > b > 0$, this expression achieves a maximum at $t = \pi/2$ and minima at $t = 0$ and $t = \pi$. Hence, the curvature is maximum when $t = 0$ or $t = \pi$ and minimum at $t = \pi/2$.