Math 3153 - Exam I. Practice Questions.

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1. Show that the midpoints of the sides of a quadilateral in $\mathbb{R}^n$ form a paralellogram.

2. Find a parametrization γ of the upper half of the unit circle, with the property that $\gamma(0) = (1, 0)$ and $\gamma(1) = (-1, 0)$

3. Find a parametrization γ of the upper half of the unit circle with the property that $\gamma(1) = (-1, 0)$ and $\gamma(1) = (1, 0)$

4. Calculate the angle between the vectors $\vec{v} = (1, -1, -2)$ and $\vec{w} = (2, 1, -1)$.

5. Determine whether the vectors $(1, -1, -2)$, $(2, 1, -1)$ and $(4, 5, 1)$, are linearly independent.

6. Consider the $\triangle ABC$ in which $A = (1, 2, 3)$, $B = (-1, -2, 2)$, and $C = (2, 3, -2)$.

   Find:

   (a) the cosine of the angle between the sides AB and AC

   (b) the length of the side AB.

   (c) the midpoint of side AB.

7. Let $\vec{\alpha} = (-1, 2, -3)$, $\quad \vec{\beta} = (3, 4, 5)$.

   (a) Evaluate $\langle \vec{\alpha}, \vec{\beta} \rangle$.

   (b) Find the projection of $\vec{\alpha}$ on $\vec{\beta}$.

   (c) Evaluate $\vec{\alpha} \times \vec{\beta}$.

8. Let $\vec{a}$, $\vec{b}$ be non zero vectors in $\mathbb{R}^3$ and such that $\vec{a} \cdot \vec{b} = 0$. Let α, β and γ are real numbers such that
$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{a} \times \vec{b} = O.$$
   Show that $\alpha = \beta = \gamma = 0$.

9. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors in 3-space. Show that
$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + \vec{b} \times \vec{c}.$$

10. Show that the diagonals of a parallelogram intersect at their midpoints.

11. Let $a = (a_1, a_2) \in \mathbb{R}^2$. Define
$$a^\perp = (-a_2, a_1)$$
    Verify that $\langle a, a^\perp \rangle = 0$.

12. Find a unit normal vector to the plane $2x - y + z = 1$.

13. Find a vector parallel to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$.

14. Find the perpendicular distance from the plane $2x - y + z = 1$ to the point $(2, 3)$.

15. Express
$$(\vec{\alpha} + 2\vec{\beta}) \times (3\vec{\alpha} + 4\vec{\beta})$$
    as a scalar multiple of $\vec{\alpha} \times \vec{\beta}$.

16. Let $\vec{a}$, $\vec{b}$ be non zero vectors in $\mathbb{R}^3$ and such that $\langle \vec{a}, \vec{b} \rangle = 0$. Let α, β and γ are real numbers such that
$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{a} \times \vec{b} = \vec{O}.$$
    Find α, β and γ.

17. Let P be a point and $\vec{u}$ a non zero vector in $\mathbb{R}^3$. Write an equation for the plane π through P with normal $\vec{u}$.

18. Find the length of the helix $(\cos 2t, \sin 2t, 3t)$ between $t = 1$ and $t = 3$.

19. Let
$$\vec{\alpha} = (3, 1, -1)$$
$$\vec{\beta} = (\lambda - 1, 7, -\frac{7}{22}\lambda)$$
    (a) Find λ if $\vec{\beta}$ and $\vec{\alpha}$ are parallel.

    (b) Find λ if $\vec{\beta}$ and $\vec{\alpha}$ are orthogonal.

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(c) Find λ if $\|\beta\| = 25$.

20. The *skew product* of two vectors in $\mathbb{R}^2$ is defined by the formula:

$$(x_1, y_1) \wedge (x_2, y_2) \;=\; x_1 y_2 - x_2 y_1$$

(a) Show that

$$(c_1 v_1) \wedge (c_2 v_2) = c_1 c_2 (v_1 \wedge v_2)$$

for any scalars c_1, c_2 and any vectors v_1, v_2.

(b) Compute

$$(\cos\alpha, \sin\alpha) \wedge (\cos\beta, \sin\beta).$$

(c) What geometric circumstance determines the sign of the skew product?

(d) Let $\vec{a}, \vec{b}$ be vectors in $\mathbb{R}^2$. Show that $\vec{a}$ and $\vec{b}$ are linearly independent if and only if $\vec{a} \wedge \vec{b} \neq 0$.

21. Let $a, b \in \mathbb{R}^3$. Show that

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a \cdot b)^2.$$

22. If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, show that $\vec{b} = \vec{c}$. Hint: See the previous question.

23. Find a unit direction vector for each of the following lines in $\mathbb{R}^2$:

(a) $2x + 3y + 4 = 0$

(b) $x = 5$

24. (a) Find a unit vector $u^\perp$ perpendicular to the line ℓ given by

$$ax + by + c = 0$$

where a and b are not both zero.

(b) Let p be the point $(0, -\frac{c}{b}) \in \ell$. and let $q = (\xi, \eta) \in \mathbb{R}^2$. Compute the projection $P_{u^\perp}(\overrightarrow{pq})$ of the free vector $\overrightarrow{pq}$ on $u^\perp$.

(c) Let $d = |P_{u^\perp}\overrightarrow{pq}|$ What does d represent geometrically?

25. Using a suitable projection, find the perpendicular distance of the point $(1, 2)$ from the line $3x + 4y + 5 = 0$.

26. Find an equation of the line incident to the point $P(3, 4, 5)$ and parallel to the line

$$\gamma(t) = (1 + t, 2 - t, 5t).$$

27. Find equations of the two planes with normal vector $(1, 1, 1)$ and tangent to the sphere $x^2 + y^2 + z^2 = 25$.

28. Let A, B be points in $\mathbb{R}^3$ and n, m non zero natural numbers. Show that the point X on the segment $\overline{AB}$ such that

$$n\|X - A\| = m\|X - B\|$$

is given by

$$X = \frac{n}{n+m} A + \frac{m}{n+m} B.$$

29. Suppose that A, B, C are three non collinear points in $\mathbb{R}^3$. Show that the medians of triangle $\triangle ABC$ intersect at the point

$$C = \frac{A + B + C}{3}.$$

30. Find a vector parallel to the line of intersection of the planes: $2x - y + z = 1$, and $3x + y + z = 2$. Find an equation for the line of intersection of the two planes.

31. Find the distance of the plane

$$2x - y + z = 1$$

from the origin.

32. Find an equation of the plane normal to the curve $\gamma(t) = (e^t, t, t^2)$ at the point $t = 1$.

33. Evaluate $\gamma'(t)$, if

$$\gamma(t) = \left(\arctan t, \ln(t^2 + 1), \arcsin t\right).$$

34. Let $\vec{\gamma}(t) = (\cos 3t, t, -\sin(3t))$.

    (a) Find the unit tangent vector $T(t)$.

    (b) Find the unit normal vector $N(t)$.

    (c) Find, in terms of x, y, and z , the equation of the osculating plane when $t = \frac{\pi}{6}$

35. Find the radius of curvature of the curve $y = 4\sin 2x$ at the point $(\pi/4, 4)$.

36. Find the tangent line to the curve $\gamma(t) = (\cos 4t, \sin 4t, t)$ at the point $t = \pi/8$.

37. Find an equation of the line normal to the curve $\gamma(t) = (\cos 3t, \sin 3t)$ at the point $t = \pi/3$.

38. (a) Let P be a point and $\vec{n}$ a non zero vector in $\mathbb{R}^3$. Write an equation for the plane π through P with normal $\vec{n}$.

    (b) Let P be a point and $\vec{n}$ a non zero vector in $\mathbb{R}^3$. Let M, N be two distinct points in the plane π through P with normal $\vec{n}$. Show that line through M and N is contained in the plane π.

39. Let α be the curve given by

$$\alpha(t) = (2\cos 3t, 2\sin 3t), \quad 0 < t < 2\pi.$$

Verify that α, α', α'' are mutually orthogonal. and find:

    (a) The unit tangent vector $T(t)$ at $t = \pi/3$.

    (b) The unit normal vector $N(t)$ at $t = \pi/3$.

    (c) The speed of α at $t = \pi/3$.

    (d) A unit speed curve β with the same image as α.

    (e) The curvature $\kappa(t)$ of β.

40. Let $\gamma(t) = (r\cos t, r\sin t, ct)$ where $r > 0$ and c are constant. Find a unit speed reparametrization of γ and compute the Frenet apparatus $[T, N, B, k, \tau]$

41. Find a differentiable curve whose image is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Where does the ellipse have maximum and minimum curvature?

42. Find the lengths of the following curves:

    (a) $\gamma(t) = (t, 2t, t^2)$; $\quad 1 \le t \le 3$

    (b) $\gamma(t) = (t^2, t^3)$; $\quad 0 \le t \le 1$.

    (c) $\gamma(t) = (e^t, e^{-t})$; $\quad 0 \le t \le 1$.

    (d) $\gamma(t) = (\cos 2t, \sin 2t, 3t)$; $\quad 1 \le t \le 3$.

43. Parametrize the graph of the function $f(x) = \cosh(x)$; $\quad 0 \le x \le \log 2$. and find the arclength.

44. Prove that if the acceleration of a curve is always perpendicular to its velocity, then its speed is constant.

45. Let $\gamma(t)$, $\beta(t)$ be two differentiable curves with the same domain. Show that

$$(\gamma \times \beta)' = \gamma \times \beta' + \gamma' \times \beta$$

46. Let $\gamma(t) = \left(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2}, 1\right)$. Show that the cosine of the angle between $\gamma(t)$ and $\gamma'(t)$ is constant.

47. Find the curvature of

    (a) The parabola $y = x^2$.

    (b) The cycloid

$$\gamma(t) = (t - \sin t, 1 - \cos t).$$

    (c) The graph of $f(t) = \cosh(t)$.

    (d) The curve $\gamma(t) = \left(\cos t, \dfrac{\cos 2t}{4}\right)$