

Terminal Velocity

Philip Pennance¹ – Version: November 13, 2023.

We study the terminal velocity of a point mass m subject to a constant gravitational force mg and a drag $-kv^2$ proportional to the square of the velocity. Note that we have chosen the direction of the vertical axis to coincide with the direction of the field.

1. The case $v_0 \geq 0$.

Newton's Law yields

$$\begin{aligned} mv' &= mg - kv^2 \\ v(t_0) &= v_0. \end{aligned}$$

We examine first the special case $m = k = g = 1$. (See [article on non dimensionalization](#).)

$$\begin{aligned} v' &= 1 - v^2 \\ v(t_0) &= v_0. \end{aligned}$$

If $v_0 = 1$, the constant function

$$v(t) = 1, \quad t \in \mathbb{R}$$

is a solution. The value $v_0 = 1$ is called the terminal velocity.

If $v_0 \neq 1$ then $v = \phi(t)$ is a solution if

$$\phi'(t) = 1 - \phi(t)^2.$$

Thus, providing $\phi(t) \neq 0$,

$$\int_{t_0}^t 1 dt = \int_{t_0}^t \frac{\phi'(t)}{1 - \phi(t)^2} dt.$$

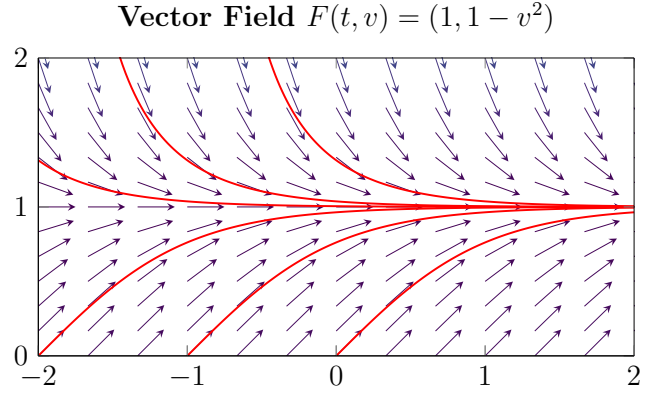
By the change of variables formula

$$\begin{aligned} t - t_0 &= \int_{\phi(t_0)}^{\phi(t)} \frac{1}{1 - v^2} dv \\ &= \begin{cases} \left. \tanh^{-1} \right|_{v_0}^{\phi(t)} & \text{if } 0 < v_0 < 1, \\ \left. \coth^{-1} \right|_{v_0}^{\phi(t)} & \text{if } v_0 > 1. \end{cases} \end{aligned}$$

It follows that

$$v(t) = \begin{cases} \tanh(t - t_0 + \tanh^{-1} v_0); & \text{if } 0 < v_0 < 1, \\ \coth(t - t_0 + \coth^{-1} v_0); & \text{if } v_0 > 1 \end{cases}$$

Typical solutions are drawn on the direction field:



2. The case $v_0 < 0$.

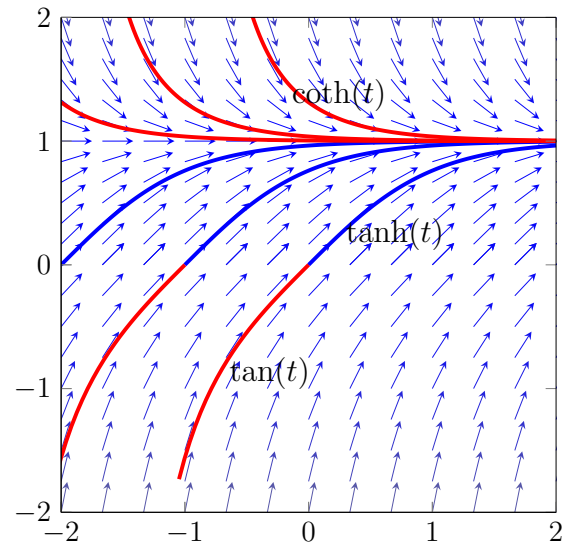
If $v_0 < 0$, the drag is in the direction of the field and the initial value problem becomes

$$\begin{aligned} v' &= 1 + v^2 \\ v(t_0) &= v_0. \end{aligned}$$

In this case, while $v < 0$,

$$v(t) = \tan(t - t_0 + \tan^{-1} v_0).$$

Vector Fields $(1, 1 - v^2)$ and $(1, 1 + v^2)$



¹<https://pennance.us>