Math 3152 - Rational Functions

Prof. Philip Pennance[1] –March 16, 2020

1. Definition.

   Let F be a field. A function $p : F \to F$ is called a *polynomial* over F if there exist $a_0, a_1, \dots, a_n \in F$ such that

   $$p(x) = a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n$$

2. Notation.

   The set of all polynomials over F will be denoted $F[x]$. Henceforth F will be either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Thus, for example, $Q[x]$ will denote the set of polynomials with rational coefficients.

3. Definition.
   A polinomial $p \in F[x]$ has *degree* n if there exist $a_0, a_1, \dots, a_n \in F$, with $a_n \neq 0$ such that

   $$p(x) = a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n.$$

   The zero polynomial has degree $-\infty$.

4. Remark.

   It is assumed that the student is familiar with the usual addition $(+)$ and multipliction (or convolution) $(*)$ of polynomials.

5. Let $f, g \in F[x]$ where F is a field. Then,

   (a) $d(f + g) \leq \max\{d(f), d(g).\}$

   (b) $d(f * g) = d(f) + d(g)$.

6. Claim. [omit]

   Let F be field. Then $(F[x], +, *)$ is a ring. I.e.,

   (a) $(F[x], +, *)$ is an abelian group.

   (b) $*$ associative.

   (c) $f * (g + h) = f * g + f * h$

   (d) $(g + h) * f = g * f + h * f$

7. Division Theorem.

   Let F be a field and let $f, g \in F[x]$, with $g \neq 0$. Then there exist unique polynomials $q, r \in F[x]$ such that $f = qg + r$ and either $r = 0$ or $d(r) < d(g)$ (and so $F[x]$ is a Euclidean domain.)

8. Special cases of the Division Theorem:

   (a) Remainder Theorem.
   Let $f \in \mathbb{C}[x]$ and $g(x) = x - a$ where $a \in \mathbb{C}$, then:

   $$f(x) = q(x)(x - a) + r$$

   where r is a constant. Moreover $r = f(a)$.

   (b) Factor Theorem.
   Let $f \in \mathbb{C}[x]$ and $a \in \mathbb{C}$. Then the following are equivalent:

   i. $x - a$ is a factor of f.

   ii. $f(a) = 0$.

9. Fundamental Theorem of Algebra
   Let $f \in \mathbb{C}[x]$ with $\deg f \geq 1$. Then f has a root $\alpha \in \mathbb{C}$.

10. Corollary.
    If $f \in \mathbb{C}[x]$ has degree $n > 0$, then f has a factorization of the form

    $$\begin{aligned} f(x) &= c(x - w_1)(x - w_2) \cdots (x - w_n) \\ &= c \cdot \prod_{j=1}^{n} (x - w_i). \qquad (1) \end{aligned}$$

    where $c, w_1, w_2, \dots, w_n \in \mathbb{C}$.

    Proof (By induction).
    If $\deg f = 1$ the result is trivial. Let $n \in \mathbb{N}$ and assume that the result holds for all polynomials of degree greater than zero but less than n. Let f be a polynomial of degree n. By the fundamental theorem of algebra $\exists \alpha \in \mathbb{C}$ such

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that $f(\alpha) = 0$. By the division theorem, $x - \alpha)$ is a factor and there exists a polynomial q such that

$$f(x) = (x - \alpha)q(x).$$

Since $\deg q = n - 1$, by the induction hypothesis, it has the form

$$c(x - w_1)(x - w_2)\cdots(x - w_{n-1}).$$

It follows that

$$f = c \cdot \prod_{i=1}^{n}(x - w_i).$$

where $w_n = \alpha$.

11. Remark.

    In the case $n = 2$ the factorization guaranteed by the Fundamental Theorem is found by completing the square. For example:

$$\begin{aligned} x^2 - 4x + 5 &= (x - 2)^2 + 1 \\ &= (x - 2)^2 - i^2 \\ &= (x - 2 - i)(x - 2 + i). \end{aligned}$$

    Notice that this factorization has the form

$$(x - z)(x - \bar{z})$$

    where $z = 2 + i$. This is not an accident but rather is a consequence of the following claim.

12. Claim.

    Let $f \in \mathbb{R}[x]$ and $z \in \mathbb{C}$. Then:

$$f(z) = 0 \Rightarrow f(\bar{z}) = 0.$$

    Thus, if a polynomial has real coefficients and $z = \alpha + \beta i$ is a root, then so is the conjugate $\bar{z} = \alpha - \beta i$

13. Corollary.

    Let $p \in \mathbb{R}[x]$ be a polynomial of degree n with real coefficients. If p has m real roots $r_1, r_2 \ldots, r_m$ and $2l$ complex roots

$$(\alpha_1 \pm i\beta_1), (\alpha_2 \pm i\beta_2), \ldots (\alpha_l \pm i\beta_l)$$

    then p has factorisation of the form:

$$c \cdot \prod_{j=1}^{m}(x - r_i) \cdot \prod_{j=1}^{l}\left[(x - \alpha_j)^2 + \beta_j^2)\right]$$

    Proof.

    By the fundamental theorem of algebra, p must have the form (1). If $z = \alpha + i\beta$ is a complex root of p then so to is $\bar{z}$. Hence, by the factor theorem, p has a factor $(x - z)(x - \bar{z})$. But

$$\begin{aligned} (x - z)(x - \bar{z}) &= x^2 - (z + \bar{z})x + z\bar{z} \\ &= x^2 - 2\alpha x + \alpha^2 + \beta^2 \\ &= (x - \alpha)^2 + \beta^2) \end{aligned}$$

    The result follows by applying this observation to every conjugate pair of roots.

14. Remark.

    It is seen from the preceeding proof that if $z \in \mathbb{C}$, then

$$(x - z)(x - \bar{z}) = x^2 - 2\mathrm{Re}(z)x + |z^2|^2.$$

15. Example.

    Let $z = 2 + i$. Then $\mathrm{Re}(z) = 2$ and $|z|^2 = 5$, and so

$$(x - z)(x - \bar{z}) = x^2 - 4x + 5.$$

16. Definition.

    A function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

    where $p(x)$ and $q(x)$ are polynomials is called a *rational function.*

17. For purposes of integration, we interested in the case $\deg p < \deg q$. This can always be achieved by polynomial division:

$$\begin{aligned}\frac{x^2+x}{x-1} &= \frac{x^2-x+2x}{x-1}\\ &= x+\frac{2x}{x-1}\\ &= x+\frac{2x-2+2}{x-1}\\ &= (x+2)+\frac{2}{x-1}\end{aligned}$$

18. By cancellation, we can assume that p and q have no common factors (and, hence, no common zeros). If all common zeros (if any) have been cancelled, any remaining zero of $q(x)$ is called a *singularity* or *pole* of the rational expresion.

19. Partial Fraction Decomposition
Let $p, q \in \mathbb{R}[x]$ with $\deg p < \deg q$. Then $\frac{p}{q}$ is a sum of terms of the following forms:

    For each factor $(x-w)$ of q with multiplicity m, there is a term:

$$\frac{A_1}{x-w}+\frac{A_2}{(x-w)^2}+\cdots+\frac{A_m}{(x-w)^m}.$$

    For each factor $(x-\alpha)^2+\beta^2)$ of q with multiplicity r there is a term:

$$\sum_{i=1}^{r}\frac{B_i x+C_i}{(x-\alpha)^2+{\beta_i}^2}.$$

20. Example.
If $q(x)=(x-2)^3[(x+1)^2+2]^2$, then the exist constants A_1, A_2, A_3, B_1, C_1, B_2, C_2 such that

$$\begin{aligned}\frac{1}{q} &= \frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-3)^3}\\ &\quad+\frac{B_1x+C_1}{(x+1)^2+2}+\frac{B_2x+C_2}{[(x+1)^2+2]^2}\end{aligned}$$

21. Example

$$\begin{aligned}\frac{3}{x^3+1} &= \frac{3}{(x+1)(x^2-x+1)}\\ &= \frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\\ &= \frac{(x+1)(Bx+C)+(x^2-x+1)A}{(x+1)(x^2-x+1)}\\ &= \frac{x^2(A+B)+x(-A+B+C)+A+C}{(x+1)(x^2-x+1)}\end{aligned}$$

Since the denominators are equal the numerators can be equated. Hence

$$\begin{cases} A+B=0\\ -A+B+C=0\\ A+C=3\end{cases}$$

Hence $A=1, \quad B=-1, \quad C=2$ and so

$$\frac{3}{x^3+1}=\frac{1}{x+1}-\frac{x-2}{x^2-x+1}$$

22. Example.
Let $p(x)=x$ and $q(x)=(x-1)^4$. Then

$$\begin{aligned}\frac{p}{q} &= \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}\\ &\quad+\frac{A_3}{(x-1)^3}+\frac{A_4}{(x-1)^4}\end{aligned}$$

Multiplying by q gives:

$$x=A_1(x-1)^3+A_2(x-1)^2+A_3(x-1)+A_4$$

Substituting $x=1$ shows that $A_4=1$. Differentiating yields

$$1=3A_1(x-1)^2+2A_2(x-1)+A_3.$$

Substituting $x=1$ shows that $A_3=1$. Differentiating again:

$$0=3\cdot 2A_1(x-1)+2A_2$$

Substituting $x=1$ shows that $A_2=0$. Differentiating yet again:

$$0=3\cdot 2\cdot A_1$$

Hence $A_1=0$. It follows that

$$\frac{x}{(x-1)^4}=\frac{1}{(x-1)^3}+\frac{1}{(x-1)^4}$$

23. Claim.

Suppose that $q \in \mathbb{R}[x]$ is a polynomial of degree n, with distinct non zero roots $w_i, \quad 1 \leq i \leq n, \quad w_i \in \mathbb{C}$ so that

$$q = \lambda(x - w_1)(x - w_2)\dots(x - w_n)$$

then there exist constants $A_1, \cdots, A_n$ such that

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - w_1} + \frac{A_2}{x - w_2} + \cdots + \frac{A_n}{x - w_n}$$

Proof.

We seek $A_1, A_2, \dots, A_n$ such that:

$$\begin{aligned} p &= \frac{A_1 q(x)}{x - w_1} + \frac{A_2 q(x)}{x - w_2} + \cdots + \frac{A_n q(x)}{x - w_n} \\ &= A_1 \lambda (x - w_2)(x - w_3)\dots(x - w_n) \\ &\quad + \frac{A_2 q}{x - w_2} + \cdots + \frac{A_n q}{x - w_n} \end{aligned}$$

Since $q(w_1) = 0$, putting $x = w_1$ yields:

$$p(w_1) = A_1 \lambda (w_1 - w_2)(w_1 - w_3)\dots(w_1 - w_n)$$

and so

$$A_1 = \frac{p(w_1)}{\prod_{j \neq 1}^{m} (w_1 - w_j)} \qquad (2)$$

It is left as a simple exercise in differentiation to show that

$$A_1 = \frac{p(w_1)}{q'(w_1)}$$

The same argument shows that:

$$A_i = \frac{p(w_i)}{q'(w_i)} \quad i = 2, ..., n \qquad (3)$$

Conversely, it is easy to verify that the A_i obtained satisfy the required partial fraction decomposition.

24. Example using (2) .

$$\frac{2x^2 + x + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A_1}{x - 1} + \frac{A_2}{x - 2} + \frac{A_3}{x - 3}.$$

Using (2):

$$\begin{aligned} A_1 &= \left. \frac{2x^2 + x + 1}{(x - 2)(x - 3)} \right|_{x=1} = 2, \\ A_2 &= \left. \frac{2x^2 + x + 1}{(x - 1)(x - 3)} \right|_{x=2} = -11, \\ A_3 &= \left. \frac{2x^2 + x + 1}{(x - 1)(x - 2)} \right|_{x=3} = 11, \end{aligned}$$

25. Example using (3) .

$$\begin{aligned} \frac{x + 2}{x^2 + 4x + 3} &= \frac{x + 2}{(x + 3)(x + 1)} \\ &= \frac{A_1}{x + 3} + \frac{A_2}{x + 1} \end{aligned}$$

Let

$$p(x) = x + 2, \quad q(x) = x^2 + 4x + 3.$$

Then

$$q'(x) = 2x + 4.$$

Hence, using (3)

$$A_1 = \frac{p(-3)}{q'(-3)} = \frac{1}{2}, \quad A_2 = \frac{p(-1)}{q'(-1)} = \frac{1}{2}$$

and so

$$\frac{x + 2}{x^2 + 4x + 3} = \frac{\frac{1}{2}}{x + 3} + \frac{\frac{1}{2}}{x + 1}$$

26. Example:

$$\begin{aligned} \frac{3}{x^3 + 1} &= \frac{3}{(x + 1)(x^2 - x + 1)} \\ &= \frac{3}{(x + 1)(x - \omega)(x - \overline{\omega})} \\ &= \frac{A_1}{x + 1} + \frac{A_2}{x - \omega} + \frac{A_3}{x - \bar{\omega}} \end{aligned}$$

where ω is a complex root of -1. Let $p(x) = 3$ and $q(x) = x^3$,so that

$$q'(x) = 3x^2.$$

Then, by (3), $\quad A_1 = \dfrac{p(-1)}{q'(-1)} = 1,$

$$A_2 = \frac{p(\omega)}{q'(\omega)} = \frac{1}{\omega^2} = -\omega$$

Similarly, $A_3 = -\bar{\omega}$.

Hence

$$\begin{aligned} \frac{3}{x^3 + 1} &= \frac{1}{x + 1} - \frac{\omega}{x - \omega} - \frac{\bar{\omega}}{x - \bar{\omega}} \\ &= \frac{1}{x + 1} - \frac{(\omega + \bar{\omega})x + 2\omega\bar{\omega}}{x^2 - x + 1} \\ &= \frac{1}{x + 1} - \frac{x - 2}{x^2 - x + 1} \end{aligned}$$

(Recall that in a quadratic $x^2 + bx + c$, The product of the roots is c and their sum is $-b$.)

Exercises– Rational Functions

Prof. Philip Pennance [2] Version: – February 2019.

1. Let
$$p(x) = 2x^2 + x + 1$$
$$q(x) = (x-1)(x-2)(x-3)$$
Find A_1, A_2, A_3 such that
$$\frac{p(x)}{q(x)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

2. Find the partial fraction expansion of:
   (a) $\dfrac{x^2}{1+x}$
   (b) $\dfrac{1}{x^2-1}$
   (c) $\dfrac{x+1}{x^2+5x+6}$
   (d) $\dfrac{x^2+2x+3}{(x-1)^2(x-4)}$
   (e) $\dfrac{1}{(x-a)(x-b)(x-c)}$ if $a \neq b \neq c$.
   (f) $\dfrac{rx+s}{(x+a)(x+b)}$ if $a \neq b$.
   (g) $\dfrac{3x^2+2x+1}{(x-1)(x-2)^2(1+x+x^2)}$

3. Let $z, w \in \mathbb{C}$. Show that:
   (a) $\overline{(z+w)} = \overline{z} + \overline{w}$.
   (b) $\overline{z^n} = (\overline{z})^n$ for all $n \in \mathbb{N}$.

4. Let p be a polynomial with real coefficients. Show that if $w \in \mathbb{C}$ is a root of p then so is $\overline{w}$.

5. Let $p \in \mathbb{R}[x]$ be a polynomial with real coefficients. Show that if $\beta + i\gamma$ is a root of p, where β and γ are real, then $(x-\beta)^2 + \gamma^2$ is a factor.

6. Use the substitution $t = \tan\left(\dfrac{x}{2}\right)$ to evaluate $\displaystyle\int \sec x \, dx$.

7. A polynomial $q \in \mathbb{R}[x]$ is of the lowest degree possible such that $q(2) = q(1+i) = 0$, $q(x) \geq 0$ for all $x \in \mathbb{R}$ and $q(1) = 1$.
   (a) Find q.
   (b) Find the partial fraction expansion of $1/q$.
   (c) Find $\displaystyle\int \frac{1}{q(x)} dx$.

8. Evaluate
   (a) $\displaystyle\int \frac{1000}{x^2+2x} \, dx$
   (b) $\displaystyle\int \frac{x+3}{x^2+2x+2} \, dx$
   (c) $\displaystyle\int \frac{x^3}{x^2+1} \, dx$
   (d) $\displaystyle\int \frac{x^8}{x^4-1} \, dx$
   (e) $\displaystyle\int \frac{x-1}{(x^2+2x+5)^3} \, dx$
   (f) $\displaystyle\int \frac{1}{x^3-x} \, dx$
   (g) $\displaystyle\int \frac{1}{x^2+2x+5} \, dx$
   (h) $\displaystyle\int \frac{1}{x^3+27} \, dx$
   (i) $\displaystyle\int \frac{x}{x^2+x+1} \, dx$

9. Integrate by reduction to rational functions:
   (a) $\displaystyle\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x} \, dx$
   (b) $\displaystyle\int \frac{\sin x}{\cos^2 x - 3\cos x} \, dx$
   (c) $\displaystyle\int \frac{\cosh t}{\sinh^2 t + \sinh^4 t} \, dt.$

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