Math 3152 - The Integration of Rational Functions

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1. Partial Fraction Expansion.

   Let $p, q \in \mathbb{R}[x]$ with $\deg p < \deg q$. Recall that $\frac{p}{q}$ is a sum of terms of the following forms:

   For each factor $(x-r)$ of q with multiplicity m, there is a term

   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

   For each factor $(x-\alpha)^2+\beta^2)$ of q with multiplicity r there is a term

   $$\sum_{i=1}^{r} \frac{B_i x + C_i}{(x-\alpha)^2 + {\beta_i}^2}.$$

   Let us call the two kinds of term type I and type II. Both types can be integrated.

2. Integration of Type I Terms.

   If $n = 1$

   $$\int \frac{1}{x-r} dx = \ln(x-r) + C.$$

   If $n > 1$,

   $$\int \frac{1}{(x-r)^n} dx = \frac{-1}{(n-1)(x-r)^{n-1}} + C.$$

3. Integration of Type I Terms.

   Let

   $$I = \int \frac{Bx + C}{[(x-\beta)^2 + \gamma^2]^n} dx.$$

   Notice

   $$\begin{aligned} I &= B \int \frac{x + \frac{C}{B}}{[(x-\beta)^2+\gamma^2]^n} dx \\ &= B \int \frac{(x-\beta) + \beta + \frac{C}{B}}{[(x-\beta)^2+\gamma^2]^n} dx \\ &= B \int \frac{x-\beta}{[(x-\beta)^2+\gamma^2]^n} dx \\ &+ (\beta B + C) \int \frac{dx}{[(x-\beta)^2+\gamma^2]^n} dx. \end{aligned}$$

The first integral in the preceeding sum is easily evaluated by the substitution $u = (x-\beta)^2 + \gamma^2$.

The second integral $\int \frac{dx}{[(x-\beta)^2+\gamma^2]^n} dx$ can be written as

$$\frac{1}{\gamma^{2n}} \int \frac{1}{\left[\left(\frac{x-\beta}{\gamma}\right)^2 + 1\right]^n} dx.$$

The substitution $u = \frac{x-\beta}{\gamma}$ reduces this to

$$\frac{1}{\gamma^{2n-1}} \int \frac{1}{(1+u^2)^n} dx.$$

Let

$$I_n = \int \frac{1}{(1+u^2)^n} du.$$

Then $I_1 = \tan^{-1} u$. If $n = 2$ then

$$\begin{aligned} I_2 &= \int \frac{(1+u^2) - u^2}{(1+u^2)^2} du \\ &= I_1 - \int u \cdot \frac{u}{(1+u^2)^2} du \\ &= I_1 + \frac{1}{2} \int u \cdot \left(\frac{1}{1+u^2}\right)' du \\ &= I_1 + \tfrac{1}{2} \cdot \frac{u}{1+u^2} - \tfrac{1}{2} I_1 \\ &= \tfrac{1}{2} \cdot \frac{u}{1+u^2} + \tfrac{1}{2} I_1 + C. \end{aligned}$$

For $n > 2$, a similar integration by parts (exercise) leads to the recursion formula

$$I_n = \frac{1}{2(n-1)} \frac{u}{[1+u^2]^{n-1}} + \frac{2n-3}{2n-2} I_{n-1}.$$

It follows that any partial fraction expansion can be integrated.

[1] https://pennance.us