Power Series

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1. Definition.

   A *power series* is a formal sum

   $$\sum_{n=0}^{\infty} a_n(x-a)^n$$

   where the coefficients a_n and also a are elements in some suitable algebraic structure. A *generating function* is a power series whose coefficients are natural numbers.

2. Claim.

   If a power series has real or complex coefficients, so that the notion of convergence makes sense, then either it

   (a) Converges only when $x = a$.

   (b) Converges for all $x \in \mathbb{R}$.

   (c) There exists a number $R > 0$, called the *radius of convergence*, such that it converges in an interval

   $$|x-a| < R$$

   and diverges if

   $$|x-a| > R.$$

   Proof.

   Notice that (a) is true for the series $\sum_{n=0}^{\infty} n!x^n$ and that (b) is true for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Thus, suppose that (a), (b) are false. Then $\sum_{n=1}^{\infty} a_n(x-a)^n$ converges at some point α with $|\alpha - a| = r > 0$. Since the terms of a convergent sequence must approach 0 there exists N such that $|a_n||\alpha - a|^n < 1$ for all $n > N$. Then

   $$\begin{aligned} |a_n||x-a|^n &= |a_n||\alpha-a|^n \left|\frac{x-a}{\alpha-a}\right|^n \\ &\le \left|\frac{x-a}{\alpha-a}\right|^n \\ &= \left|\frac{x-a}{r}\right|^n \end{aligned}$$

   By comparison with the geometric series, $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for all x with $|x-a| < r$. Let R to be the supremum of the set

   $$\left\{ |x-a| : \sum_{n=1}^{\infty} a_n(x-a)^n \text{ converges} \right\}$$

   Then, clearly, the series converges for all x in the interval $|x-a| < R$ and diverges if $|x-a| > R$.

3. Remark.

   It is convenient to define $R = 0$ in case the series converges only at the single point $x = a$ and $R = \infty$ if convergence occurs for all x. If a is complex, the interval of convergence becomes a disc.

4. Example.

   Test the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

   Solution.

   $$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2n+3}/(2n+3)}{(-1)^n x^{2n+1}/2n+1}\right| \\ &= |x^2| \left|\frac{2n+1}{2n+3}\right|. \end{aligned}$$

   $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x^2|.$$

   Since $|x^2| < 1$ if and only if $|x| < 1$ the radius of convergence $R = 1$.

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5. Example.

   Find the radius of convergence of the series
$$\sum_{n=0}^{\infty} \frac{x^n n^2}{2^n(n+1)}$$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{x^{n+1}}{2^{n+1}}\frac{(n+1)^2}{(n+2)}}{\frac{x^n n^2}{2^n(n+1)}}\right| \\ &= \frac{|x|}{2}\lim_{n\to\infty}\left|\frac{(n+1)^3}{n^2(n+2)}\right|\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2}$$

   Since $\frac{|x|}{2} < 1$ iff $|x| < 2$ the radius of convergence $R = 2$.

6. Example

$$f(x) = \sum_{n=0}^{\infty} \frac{n^n}{n!} x^n$$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{|x|}{n+1} \\ &= \left(1+\frac{1}{n}\right)^n |x|\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e|x|$$

   By the ratio test the series converges absolutely provided

$$|x| < 1/e$$

7. Definition.

   Let f be a function posessing derivatives of all orders. The power series

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

   is called the *Taylor Series* of f about the point a. If $a = 0$ the series is called a *Maclaurin series*

8. Find the Maclaurin series of the polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

   Solution.

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \\ f''(x) &= 2\cdot 1 a_2 + 3\cdot 2 a_3 x \\ f^{(3)}(x) &= 3\cdot 2\cdot 1 a_3 \\ f^{(n)}(x) &= 0, \quad n > 3 \end{aligned}$$

   The Maclaurin series $T_{f,0}(x)$ is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + 0x^4 + 0x^5 + \cdots$$

   Hence, the Maclaurin series $T_{f,0}$ converges to f. This is a special case of the following claim.

9. Claim.

   Let
$$f(x) = \sum_{n=1}^{n} a_i x^i$$
   be a polynomial of degree n. Then, its degree n Taylor polynomial T_n is just the same polynomial f.

   Proof.

   It is an easy induction (exercise) to verify that
$$\frac{f^{(i)}(0)}{i!} = a_i \tag{1}$$
   from which the result follows.

10. Taylor's Theorem

   Suppose that f has $n+1$ continuous derivatives on an open interval containing a. Then, for x in this interval

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n. \tag{2}$$

   where

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \tag{3}$$

Proof.

By induction on n. If $n = 0$ then (2) is the statement

$$f(x) = f(a) + \int_a^x f'(t)dt$$

which is true by the Fundamental Theorem. Let $n \in \mathbb{N}$ and assume that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_{n-1}. \tag{4}$$

Then

$$\begin{aligned} R_{n-1} &= \int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}dt \\ &= -\int_a^x \frac{f^{(n)}(t)}{(n-1)!}\frac{d}{dt}\frac{(x-t)^n}{n}dt \\ &= -f^{(n)}(t)\frac{(x-t)^n}{n!}\bigg|_a^x + \int_a^x \frac{f^{(n)}(t)}{n!}(x-t)^n dt \\ &= \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n. \end{aligned}$$

Substituting for R_{n-1} in (4) yields

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k + R_n.$$

It follows, by induction, that the theorem is true for all natural numbers n.

11. The expression for R_n in Taylor's theorem is called the *integral form of the remainder*

12. Example.

    Let $f(x) = e^x$. Then $f^{(k)}(0) = 1$, for all k. Hence Taylor's theorem with $a = 0$ gives

    $$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x).$$

    The remainder depends on x. If $x = 1$, then

    $$e = \sum_{k=0}^{n} \frac{1}{k!} + R_n$$

    and

    $$\begin{aligned} R_n &= \int_0^1 \frac{1}{n!}e^t(1-t)^n\,dt \\ &\le \frac{e}{n!}\int_0^1 (1-t)^n\,dt \\ &= \frac{e}{(n+1)!.} \end{aligned}$$

13. Corollary.

    e is not rational.

    Proof.

    Suppose to the contrary that there exist natural numbers a, b such that

    $$\begin{aligned} \frac{a}{b} &= e \\ &= \sum_{k=0}^{n} \frac{1}{k!} + R_n \end{aligned}$$

    If n is chosen so that $n > b$, then $\frac{n!}{b}b$ and $\frac{n!}{k!}$ are both natural numbers. Therefore

    $$n!R_n = \frac{n!a}{b} + \sum_{k=0}^{n} \frac{n!}{k!} \in \mathbb{N}$$

    On the other hand, if $n > 3$ then

    $$n!R_n < \frac{e}{n+1} < 1$$

    a contradiction. It must be that e is irrational.

14. Claim.

    The integral form of the remainder in Taylor's theorem satisfies

    $$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0. \tag{5}$$

    Proof.

    Since f^{n+1} is continuous, it achieves a maximum M on $[a, x]$. Hence,

$$\begin{aligned} R_n(x) &= \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt \\ |R_n(x)| &\le \int_a^x \frac{|f^{(n+1)}(t)|}{n!}|x-a|^n\,dt \\ &= \frac{|x-a|^n}{n!}\int_a^x |f^{(n+1)}(t)|\,dt \\ &\le \frac{|x-a|^n}{n!} M \cdot |x-a| \end{aligned}$$

It follows that

$$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0.$$

15. Claim. (Lagrange form of the remainder.)

    There exists ξ in an open interval containing a such that

    $$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

    Proof.

    Follows from the Cauchy mean value theorem for integrals. See text.

    Remark.

    Putting $x = a + h$ in the Taylor series about a

    $$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \cdots$$

    yields the alternative form

    $$f(a+h) = f(a) + f'(a)\cdot h + \frac{f''(a)}{2!}\cdot h^2 + \cdots$$

    Using the Lagrange form of the remainder, this means for example that

    $$f(x+h) = f(x) + h\cdot f'(x) + \frac{h^2}{2!}\cdot f''(\xi)$$

    for some $\xi \in [x, x+h]$.

16. Example.

    Use a Taylor polynomial to estimate $\sinh(1)$.

    Solution.

    Using the Lagrange form of the remainder yields:

    $$\begin{aligned} \sinh x &= x + \frac{x^3}{3!} + \frac{\sinh^{(4)}(\xi)}{4!} \\ &= x + \frac{x^3}{3!} + \frac{\cosh(\xi)}{4!} \end{aligned}$$

    for some ξ with $0 < \xi < x$. Now cosh is increasing on $[0, 1]$. Hence

    $$\begin{aligned} \cosh\xi &\le \cosh 1 \\ &\le \frac{e+e^{-1}}{2} \\ &\le e \\ &\le 3 \end{aligned}$$

    Hence

    $$\sinh 1 = 1 + \frac{1}{3!} = \frac{7}{6}$$

    with error $R_4 < \dfrac{3}{4!}$.

17. Remark.

    Many exam questions take the following forms:

    (a) Given an error ("tolerance") $\epsilon > 0$, find the smallest degree k such that the Taylor polynomial $T_{f,k,a}$ approximates f with error less than ϵ for all x in some specified interval.

    (b) Given an error $\epsilon > 0$ and a degree k, find the maximum radius r of an interval $x - a| < r$, centered at a, such that the Taylor polynomial $T_{f,k,a}$ approximates f with error with error less than ϵ for all x in the interval.

18. Definition.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

is called the n'th *Taylor polynomial* of f about the point a.

19. Example.

Find the first 5 Taylor polynomials of the function

$$p(x) = -1 + 3x - 3x^2 + x^3$$

about the origin.

Solution.

Using (1) yields

$$\begin{aligned} T_0 &= -1 \\ T_1 &= -1 + 3x \\ T_2 &= -1 + 3x - 3x^2 \\ T_3 &= -1 + 3x - 3x^2 + x^3 \\ T_4 &= -1 + 3x - 3x^2 + x^3 \\ T_5 &= -1 + 3x - 3x^2 + x^3. \end{aligned}$$

20. Example.

Compute the Taylor series of the function $f(x) = \ln x$ about $x = 1$.

Solution.

Notice,

$$\begin{aligned} \ln'(x) &= \frac{1}{x} \\ \ln''(x) &= -\frac{1}{x^2} \\ \ln^{(3)}(x) &= \frac{2!}{x^3} \\ \ln^{(4)}(x) &= -\frac{3!}{x^4} \\ \ln^{(5)}(x) &= \frac{4!}{x^5} \\ &\vdots \end{aligned}$$

and by induction,

$$\ln^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad n \geq 1$$

Therefore,

$$\log^{(k)}(1) = (-1)^{k-1}(k-1)!$$

and so

$$T_{f,1} = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

21. Example.

Find the Maclaurin series for

$$g(x) = \ln(x+1).$$

Solution.

Notice that the graph of g is a translation the graph of $\ln x$ by one unit to the left. It follows that the required series can be obtained from that of $\ln x$ about by translation.

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

$$\therefore \ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

Remark.

In general, if $g(x) = f(x-a)$, it is easy to verify that

$$T_{g,a}(x) = T_{f,0}(x-a)$$

22. Exercise.

Find the third degree Taylor polynomial T about the point $x = -1$ of

$$p(x) = x^3 + x^2 + x + 1.$$

23. Example.

Show that the Taylor series $T_{f,0}(x)$ for $\sin x$ converges to $\sin x$ for all $x \in \mathbb{R}$.

Solution.

It suffices to show that the remainder $R_n(x)$ converges to zero.

Using the Lagrange form of the remainder,

$$\begin{aligned}R_n(x) &= \sin x - T_n(x)\\ &= \frac{\sin^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\end{aligned}$$

for some ξ between 0 and x. Hence

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

It is a simple exercise to show that for any $x \in \mathbb{R}$

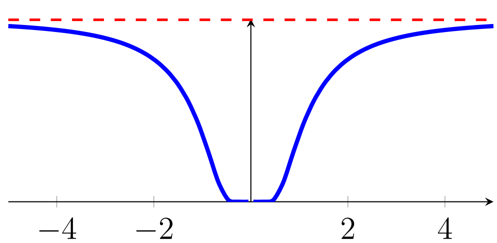
$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows that the Taylor series for $\sin x$ converges to $\sin x$ for all $x \in \mathbb{R}$

24. Remark.

A function need not equal its Taylor series. Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



which, as the drawing indicates, is very flat close to 0.

The Maclaurin series $T_{f,0}$ has all coefficients zero (Prove this). Therefore, $f \neq T_{f,0}$.

Remark.

In the complex plane the function f defined above, is not continuous at 0 since it has no limit as $z \to 0$ along the i axis.

25. Remark.

In the complex domain, a function is called *analytic* if at every point of its domain, the Taylor series $T_{f,a}$ converges to the function in a neighborhood of the point and is called *holomorphic* if, at every point of its domain, it is (complex) differentiable in a neighborhood of the point. In the complex domain, these two concepts are equivalent.

26. Examples of analytic functions.

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\\ &= 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

27.

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\end{aligned}$$

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\end{aligned}$$

$$\begin{aligned}\cosh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\\ &= 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \ldots\end{aligned}$$

$$\begin{aligned}\sinh(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\end{aligned}$$

28. Definition. (Infinitesimal asymptotic notation.)

Let f and g be two functions. Then f is of *order g as x approaches zero*, written $f \in O(g(x))$ as $x \to 0$, if there exists a positive constant M such that

$$|f(x)| < M|g(x)|$$

for all $|x|$ sufficiently small. To show that $f \in O(g(x))$ as $x \to 0$ it suffices to show that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = c$$

where $c \neq \infty$.

29. Example.

Let

$$f(x) = 2x^2 + 3x^3 + 8x^9$$

then $f \in O(x^2)$ as $x \to 0$ since

$$|2x^2 + 3x^3 + 8x^9| < 13|x|^2$$

for all $|x| < 1$

30. Example.

Let

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^{n+1}$$

be the Lagrange form of the remainder in a Maclaurin series, then

$$R_n \in O(x^{n+1}.)$$

as $x \to 0$.

31. Convention.

In the approximation of a function by a series the most significant terms are written explicitly, and the least-significant terms summarized in a single "big O" term.

32. Example.

The approximation

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3)$$

is interpreted to mean that

$$\begin{aligned} |R_2(x)| &= \left| e^x - \left(1 + x + \frac{x^2}{2}\right)\right| \\ &\leq M\left|x^3\right| \end{aligned}$$

for all x sufficiently close to 0.

33. Definition.

Two functions $f,\, g : \mathbb{R} \to \mathbb{R}$ are said to be *equal up to order n at a* if

$$\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

Notation.

If f, g are equal up to order n we write

$$f \overset{(n,a)}{\approx} g$$

34. Claim.

$\overset{(n,a)}{\approx}$ is an equivalence relation. I.e.,

(a) $f \overset{(n,a)}{\approx} f$

(b) $f \overset{(n,a)}{\approx} g \Rightarrow g \overset{(n,a)}{\approx} f$

(c) If $f \overset{(n,a)}{\approx} g$ and $g \overset{(n,a)}{\approx} h$ then $f \overset{(n,a)}{\approx} h$

Proof of transitivity:

Suppose that $f \overset{(n,a)}{\approx} g$ and $g \overset{(n,a)}{\approx} h$. Notice

$$\frac{f-h}{(x-a)^n} = \frac{f-g}{(x-a)^n} + \frac{g-h}{(x-a)^n}$$

The result follows by taking limits. Reflexivity and symmetry are even more trivial.

35. Example.

Let

$$\begin{aligned} f(x) &= e^x \\ g(x) &= 1 + x + \frac{1}{2!}x^2 \end{aligned}$$

Then it follows (using l'hospital) that

$$\lim_{x\to 0} \frac{f(x) - g(x)}{x^2} = 0$$

proving $f \overset{(2,0)}{\approx} g$. As the next result shows, this is not an accident.

36. Lemma.

If $0 \le i \le n$ then

$$f \overset{(n,a)}{\approx} g \Rightarrow f \overset{(i,a)}{\approx} g$$

Proof.

If $0 \le i \le n$ and $f \overset{(n,a)}{\approx} g$ then

$$\begin{aligned}\lim_{x\to a} \frac{f(x)-g(x)}{(x-a)^i} &= \lim_{x\to a} \frac{f(x)-g(x)}{(x-a)^n}.(x-a)^{n-i} \\ &= 0\cdot 0 \\ &= 0\end{aligned}$$

37. Lemma.

Let p, q be polynomials in $(x-a)$ of degree at most n. Then

$$p \overset{(n,a)}{\approx} q \Rightarrow p = q. \qquad (6)$$

Proof.

$p-q$ is a polynomial

$$a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$$

of degree at most n.

By the lemma

$$p \overset{(i,a)}{\approx} q, \quad 0 \le i \le n.$$

Since $p \overset{(0,a)}{\approx} q$

$$\lim_{x\to a}[p(x)-q(x)] = 0$$

and so $a_0 = 0$. Hence,

$$p - q = a_1(x-a) + \cdots + a_n(x-a)^n$$

Since $p \overset{(1,a)}{\approx} q$

$$\lim_{x\to a} \frac{p(x)-q(x)}{x-a} = 0$$

It follows that $a_1 = 0$. Proceeding in like manner (or using induction) yields $a_0 = a_1 = \cdots = a_n = 0$ and so $p = q$.

38. Theorem.

Let p be a polynomial of degree n and f a function with Taylor polynomial T_n of degree n at $x = a$. Then,

$$p \overset{(n,a)}{\approx} f \Rightarrow p = T_n \qquad (7)$$

I.e., the Taylor polynomial of degree n is the unique polynomial which is equal to f up to order n.

Proof.

Suppose $p \overset{(n,a)}{\approx} f$. The Lagrange form of the remainder shows that $f \overset{(n,a)}{\approx} T_n$. By transitivity $p \overset{(n,a)}{\approx} T_n$ It foolows using (6) that $p = T_n$.

39. Corollary.

$$f \overset{(n,a)}{\approx} g \to T_{f,n,a} = T_{g,n,a}$$

40. Example.

Let

$$g(x) = 1 + x + x^2 + x^3$$

and

$$f(x) = 1 + x + x^2$$

Since $f \overset{(2,0)}{\approx} g$ it must be that

$$f = T_{g,2,0}$$

41. Example.

Find the Maclaurin series of the function

$$f(x) = \frac{1}{1-x}.$$

Solution.

Let $x \in \mathbb{R}$ with $|x| < 1$. Then for all $n \in \mathbb{N}$, the formula for the sum of a geometric sequence yields

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1-x} \qquad (8)$$

Let
$$p(x) = \sum_{k=0}^{n} x^k.$$
Then
$$\lim_{x\to 0} \frac{f(x)-p(x)}{x^n} = \lim_{x\to 0} \frac{x}{1-x} = 0$$

Hence $f \overset{(n,0)}{\approx} p$ and it follows from (7) that p is the n'th Taylor polynomial of f and
$$R_n = \frac{x^{n+1}}{1-x}$$
–which can be verified by computing the integral form of the remainder.

42. Remark.

In algebraic terminology, we have shown that each polynomial of degree n is an equivalence class representative of the relation $\overset{(n,a)}{\approx}$.

43. Example.

Find a Maclaurin series for $\displaystyle\int \frac{\sin x}{x}\,dx$.

Solution.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \\
\frac{\sin x}{x} &= \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{x^{2n}}{(2n+1)!} \\
\int \frac{\sin x}{x} &= \int \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n+1}}{(2n+1)(2n+1)!}
\end{aligned}$$

44. Remark.

Taylor Series may be used to check critical points.) At a critical point, $f'(a) = 0$ and so the Taylor series becomes
$$f(x) = f(a) + \frac{1}{2}(x-a)^2 f''(a) + \mathcal{O}(x-a)^3$$

It follows (exercise) that:

(a) $f''(a) > 0$ indicates a local minimum.

(b) $f''(a) < 0$ indicates a local maximum.

If $f''(a) = 0$ and $f^{(3)} \neq 0$ then
$$f(x) = f(a) + \frac{1}{3!}(x-a)^3 f'''(a) + O(x-a)^4$$

In this case it is easily seen that $x = a$ is neither a maximum or minimum.

45. Example.

Find the Maclaurin series of $\log(x+1)$.

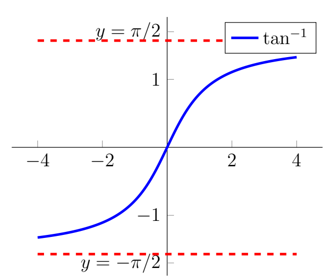
Solution.

Replacing x by $-x$ in (8) gives
$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n x^n,
\end{aligned}$$

It is legitimate to integrate a convergent Taylor series term by term. Integrating and recalling that $\log 1 = 0$ gives
$$\begin{aligned}
\log(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^6).
\end{aligned}$$

46. Example.

Find the Maclaurin series of $\tan^{-1} x$.



Solution.

Recall

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

If $|x| < 1$, we can replace x by in (8).

$$\begin{aligned}\frac{d}{dx}\arctan(x) &= \frac{1}{1-(-x^2)} \\ &= 1 - x^2 + x^4 - x^6 + \cdots\end{aligned}$$

Integrating term by term

$$\begin{aligned}\arctan(x) &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right)\, dx \\ &= \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\, dx \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C\end{aligned}$$

Putting $x = 0$ gives $C = 0$. Hence,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{8} + O(x^9)$$

47. Example.

Find the Maclaurin series of $\sin^{-1} x$

Solution.

Let $y = \sin^{-1} x$. Then

$$\begin{aligned}y' &= \frac{1}{\sqrt{1-x^2}} \\ y'' &= x(1-x^2)^{-3/2}\end{aligned}$$

Hence

$$(1-x^2)y'' = xy'$$

Differentiating n times using Leibniz rule yields the recursion:

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2 y^{(n)} = 0$$

and so

$$y^{(n+2)}(0) = n^2 y^{(n)}(0) \quad \forall n$$

Since $y(0) = 0$, all even derivatives are zero. From $y'(0) = 1$ follows

$$\begin{aligned}y^{(3)}(0) &= 1 \\ y^{(5)}(0) &= 3^2 \\ y^{(7)}(0) &= 5^2 \cdot 3^2\end{aligned}$$

Hence

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

48. Example.

Find the Maclaurin series of

$$f(x) = (1+x^2)\cosh x.$$

Solution

$$\begin{aligned}&(1+x^2)\cosh x \\ &= (1+x^2)\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{6!} + \ldots\right) \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \\ &+ \quad x^2 + \frac{x^4}{2} + O(x^6) \\ &= 1 + \frac{3x^2}{2} + \frac{13}{24}x^4 + O(x^6)\end{aligned}$$

49. Example.

Find the Maclaurin series of $\sec x$.

Solution.

Long division of power series can be used. Let

$$\begin{aligned}T &= 1 - \cos t \\ &= \frac{x^2}{2} - \frac{x^4}{4!} + \cdots\end{aligned}$$

Then

$$\begin{aligned}
\sec x &= \frac{1}{\cos x} \\
&= \frac{1}{1-T} \\
&= 1 + T + T^2 + O(T^3) \\
&= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^2}{2} - \frac{x^4}{24}\right)^2 + O(x^6) \\
&= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \left(\frac{x^2}{2}\right)^2 + O(x^6) \\
&= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots
\end{aligned}$$

50. Example.

Find the Maclaurin series of

$$f(x) = (1+x)^\alpha$$

where $\alpha \in \mathbb{R}$.

Solution.

It can be shown that

$$T_{f,0} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

where $\binom{\alpha}{0} = 1$ and for $n > 0$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

If $\alpha \in \mathbb{N}$, the power series terminates and the series becomes the binomial theorem. Otherwise, it is called the *Newton binomial series with parameter* α and converges provided $|x| < 1$.

51. Examples of Binomial Series

$$\begin{aligned}
(1+x)^{-1} &= 1 - x + x^2 - x^3 + \cdots \\
(1+x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + + \cdots \\
(1+x)^{-3} &= 1 - 3x + 6x^2 - 10x^3 + + \cdots \\
(1+x)^{-\frac{1}{2}} &= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \cdots \\
(1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \cdots \\
\left(1 + \frac{x^2}{2}\right)^{1/2} &= 1 + \frac{x^2}{4} - \frac{x^4}{32} + \cdots
\end{aligned}$$

Exercises

Prof. Philip Pennance [2]   Version: – February 2019.

For each of the following series, determine whether it converges. If so, determine the radius of convergence $R \in [0, \infty]$.

1. $\displaystyle\sum_{n=1}^{\infty} x^n$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$
3. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{4n+1}(x+1)^n$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+3}}(x-7)^n$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{4n+1}(x-2)^k$
8. $\displaystyle\sum_{n=10}^{\infty} n x^n$
9. $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1}(x+1)^n$
11. $\displaystyle\sum_{n=1}^{\infty} n x^n$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$
13. $\displaystyle\sum (-1)^n \frac{x^n}{n}$
14. $\displaystyle\sum \frac{n^n x^n}{n!}$
15. $\displaystyle\sum \frac{x^n}{n!}$

1. Find the Taylor polynomial $P_{f,a,n}(x)$ of degree $n = 3$ at $a = 1$ of the function $f(x) = x^3$.
2. Find the Taylor polynomial $P_{f,a,n}(x)$ of degree $n = 5$ at $a = 0$ of the function $f(x) = (x-1)^6$.
3. Find functions f, g with $f \neq g$ yet such that f and g have the same Taylor polynomials of degree 3 at 0.
4. Find the Taylor polynomial of degree 2 of
$$P(x) = 1 + x + x^2 + x^3$$
at $a = 0$.
5. Let
$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$
be a polynomial over $\mathbb{R}$. Show that the coefficients satisfy
$$c_k = \frac{f^k(0)}{k!}.$$
6. Find the Taylor series $T_{f,a}(x)$ centered at the point $a = 1$ of $f(x) = x^n$.
7. Find the Taylor series with center at $x = -1$, of , $f(x) = 1/x^2$.
8. Find the Taylor series $T_{f,a}(x)$ centered at the point $a = 1$ of $f(x) = e^{x-1}$.
9. Let $a = \pi/36$.
   (a) Find an upper bound for the absolute value of the error in the approximation
   $$\sin a \;\approx\; a - \frac{a^3}{3!} + \frac{a^5}{5!}$$
   (b) How many terms of the Taylor series
   $$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
   are required to approximate $\sin x$ accurate to 10 decimal places on the interval $|x| < 1$?
10. Find an upperbound for the absolute value of the error in the approximation of e^x at $x = 1$ by its Taylor polynomial of degree n.

[2] https://pennance.us

11. Find an upper bound for the remainder $R_{n,0}(x)$ when $x = 1$, in the Taylor expansion of of $f(x) = e^x$ about the origin.

12. Let
$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1}$$
It is known that f is equal up to order $2n+1$ to $\arctan x$ at the origin. Find $\arctan^{(7)}(0)$.

13. Find the Maclaurin series of:
    (a) $f(x) = \dfrac{x+1}{x-1}$.
    (b) $f(x) = \tanh^{-1} x$.
    (c) $f(x) = \sqrt{1+x}$.

14. If $g(x) = f(x-a)$, show that
$$T_{g,a}(x) = T_{f,0}(x-a)$$
i.e., the Taylor series of g about a is a translation of the Maclaurin series of f.

15. Find $\displaystyle\int_{-1}^{0} \frac{x^{n+1}}{x-1}\, dx$ by integrating both sides of the relation:
$$\frac{x^{n+1}}{x-1} = 1 + x + x^2 + \ldots + x^n + \frac{1}{x-1}$$

16. Use the fact that
$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$$
to find the Maclaurin series of the function $f : (-1, 1) \to \mathbb{R}$ given by:
$$f(x) = \log \left| \frac{1+x}{1-x} \right|.$$

17. Find the Maclaurin series of the function $f : (-1, 1) \to \mathbb{R}$ given by
$$f(x) = \int_0^x \frac{1}{t} \log\left(\frac{1}{1-t}\right) dt$$
Hint: Recall
$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

18. Write the sum of the the power series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n}.$$
in terms of the logarithm function. What is the radius of convergence?

19. Find the the MacLaurin series of $f(x) = \sin \pi x$ and determine its radius of convergence.

20. Use Taylor polynomials of order 4 to approximate $f(h)$.
    (a) $f(x) = \sqrt[4]{16+h}$; with $h = 15$
    (b) $f(x) = \cos x$; with $h = \frac{\pi}{12}$

21. Find functions f, g with $f \neq g$ yet such that f and g have the same Taylor polynomials of degree 4 at 0.

22. Find the Taylor polynomial of degree 2 for $P(x) = 1 - x + 3x^2 + x^3$ at $a = 0$.

23. The polynomial
$$P(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1}$$
is equal up to order $2n+1$ to $\arctan x$ at the origin. Find $\arctan^{(7)}(0)$.

24. Let f be a function and $g(x) = f(x+a)$. Suppose that g has Taylor polynomial $P(x) = x + 4x^3 + 10x^5$ about $x = 0$. Find the Taylor polynomial Q of f about $x = a$.