Trigonometric and Hyperbolic Functions – A Comparison

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1. Definition.

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

2. Exercise.

Let $A : [-1, 1] \to [0, \pi]$ given by

$$A(\xi) = \xi\sqrt{1-\xi^2} + 2\int_{\xi}^{1} \sqrt{1-x^2}dx$$

Show that:

(a) $A'(\xi) = \dfrac{-1}{\sqrt{1-\xi^2}}$

(b) $A(-1) = \pi$

(c) $A(0) = \pi/2$

(d) $A(1) = 0$

(e) A is a bijection.

Let $A_h : [1, \infty] \to [0, \infty)$ be given by

$$A_h(\xi) = \xi\sqrt{\xi^2-1} - 2\int_{1}^{\xi} \sqrt{x^2-1}dx$$

Show that:

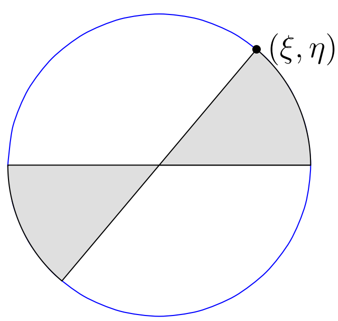
(a) $A_h'(\xi) = \dfrac{1}{\sqrt{\xi^2-1}}$

(b) $A_h(1) = 0$

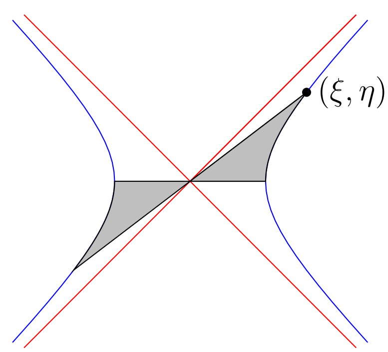
(c) A_h is a bijection.

3. Geometric interpretation.

$A(\xi)$ is the area of the double sector of the unit circle $x^2 + y^2 = 1$, determined by the point (ξ, η).



$A_h(\xi)$ is the area of the double sector of the hyperbola $x^2 - y^2 = 1$, determined by the point (ξ, η).



4. Inverse functions.

$$A^{-1} : [0, \pi] \to [-1, 1]$$

is called the *cosine* function.

$$\cos = A^{-1}$$

$$A_h^{-1} : \mathbb{R} \to [1, \infty]$$

is called the *hyperbolic cosine* function.

$$\cosh = A_h^{-1}$$

[1] https://pennance.us

5. Definition.

The *sine* function is defined by

$$\sin t = \sqrt{1-\cos^2 t}, \quad 0 \le t \le \pi$$

The *hyperbolic sine* function is defined by

$$\sinh t = \sqrt{\cosh^2 - 1}, \quad 0 \le t$$

6. Derivatives.

$$\cos' = -\sin$$

$$\cosh' = \sinh$$

Proof.

$$\begin{aligned} & A(\cos t) = t \\ \Rightarrow \quad & A'(\cos t)\cos' t = 1 \\ \Rightarrow \quad & \frac{1}{\sqrt{1-\cos^2 t}}\cos' t = 1 \\ \Rightarrow \quad & \cos' t = -\sin t \end{aligned}$$

$$\begin{aligned} & A_h(\cosh t) = t \\ \Rightarrow \quad & A_h'(\cosh t)\cosh' t = 1 \\ \Rightarrow \quad & \frac{1}{\sqrt{\cosh^2 t - 1}}\cosh' t = 1 \\ \Rightarrow \quad & \cosh' t = \sinh t \end{aligned}$$

7.

$$\sin' t = \cos t$$

Proof

$$\begin{aligned} \sin' t &= \left(\sqrt{1-\cos^2 t}\right)' \\ &= \frac{\cos t \sin t}{\sqrt{1-\cos^2 t}} \\ &= \cos t \end{aligned}$$

$$\sinh' t = \cosh t$$

Proof

$$\begin{aligned} \sinh' t &= \left(\sqrt{\cosh^2 t - 1}\right)' \\ &= \frac{\cosh t \sinh t}{\sqrt{\cosh^2 t - 1}} \\ &= \cosh t \end{aligned}$$

8. Extensions.

The domains of cos and sin are extended to all of $\mathbb{R}$ by declaring cos to be even and sin to be odd on $[-\pi, \pi]$ and both to be periodic on $\mathbb{R}$ with period 2π

The domains of cosh and sinh is extended to all of $\mathbb{R}$ by declaring *cosh* to be even and *sinh* to be odd.

9. Exercise. Using integration by substitution show that:

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \sin^{-1} x$$

$$2\int \sqrt{x^2-1}\,dx = x\sqrt{x^2-1} - \ln\left[x + \sqrt{x^2-1}\right]$$

10. Exercise. Show that:

$$A(\xi) = \tfrac{\pi}{2} + \sin^{-1}(-\xi)$$

$$A_h(\xi) = \ln\left[\xi + \sqrt{\xi^2 - 1}\right]$$

11. Exercise. Deduce from (10) that

$$\cosh t = \frac{e^t + e^{-t}}{2}.$$

Inverse Hyperbolic Functions

Philip Pennance[2] Version: – February 2019.

1. $\cosh : (0,\infty) \to (1,\infty)$ is a bijection and hence has an inverse $\cosh^{-1}$ which satisfies

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ \iff x &= \cosh^{-1} y \end{aligned}$$

2. Claim.

$$\cosh^{-1} = \ln\left(y + \sqrt{1 - y^2}\right)$$

Proof.

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ \iff & e^x + e^{-x} = 2y \\ \iff & e^{2x} + 1 = 2ye^x \\ \iff & (e^x)^2 + 2ye^x + 1 = 0 \\ \iff & (e^x - y)^2 - y^2 + 1 = 0 \\ \iff & e^x = y + \sqrt{y^2 - 1} \\ \iff & y = \ln\left(y + \sqrt{y^2 - 1}\right). \end{aligned}$$

Hence

$$\cosh^{-1} y = \ln\left(y + \sqrt{y^2 - 1}\right)$$

3. Similarly, $\sinh : \mathbb{R} \to \mathbb{R}$ is a bijection and hence has an inverse $\sinh^{-1}$ which satisfies

$$\begin{aligned} y &= \frac{e^x - e^{-x}}{2} \\ \iff x &= \sinh^{-1} y \end{aligned}$$

4. Claim.

$$\sinh^{-1} = \ln\left(y + \sqrt{1 + y^2}\right)$$

Proof.

$$\begin{aligned} y &= \frac{e^x - e^{-x}}{2} \\ \iff & e^x - e^{-x} = 2y \\ \iff & e^{2x} - 1 = 2ye^x \\ \iff & (e^x)^2 - 2ye^x - 1 = 0 \\ \iff & (e^x - y)^2 - y^2 - 1 = 0 \\ \iff & e^x = y + \sqrt{y^2 + 1} \\ \iff & y = \ln\left(y + \sqrt{y^2 + 1}\right). \end{aligned}$$

Hence

$$\sinh^{-1} y = \ln\left(y + \sqrt{y^2 + 1}\right)$$

5. Claim.

$$\begin{aligned} \sinh(x+y) &= \sinh x \cdot \cosh y + \cosh x \cdot \sinh y. \\ \cosh(x+y) &= \cosh x \cdot \cosh y + \sinh x \cdot \sinh y. \end{aligned}$$

Proof.

$$\begin{aligned} 4 \sinh x \cosh y + \cosh x \sinh y &= (e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y}) \\ &= 2e^x e^y - 2e^{-x} e^{-y} \\ &= 2\left(e^{x+y} - e^{x-y}\right) \\ &= 4 \sinh(x+y), \end{aligned}$$

The second formula is proven in a similar manner. A large number of corollaries follow from these addition formulae:

6. Corollary.

$$\begin{aligned} \cosh 2x &= \cosh^2 + \sinh^2 \\ &= 2\cosh^2 - 1 \\ &= 1 + 2\sinh^2 x. \end{aligned}$$

[2] https://pennance.us

$$\sinh 2x = 2 \sinh x \cosh x.$$

7. Definition.

   The hyperbolic tangent function $\tanh : \mathbb{R} \to (-1, 1)$ is defined by

$$\tanh x = \frac{\sinh x}{\cosh x}$$

8. Corollary.

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y}.$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

9. Definition.

   (a) *Hyperbolic cotangent*

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

   (b) *Hyperbolic secant*

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

   (c) *Hyperbolic cosecant*

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

10. Claim.

$$\begin{aligned} \cosh^2 x - \sin^2 x &= 1 \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \\ \coth^2 x - 1 &= \operatorname{csch}^2 x \end{aligned}$$

11. Claim.

    There is a unique function $f : \mathbb{C} \to \mathbb{C}$ such that for all $z, w \in \mathbb{C}$

    (a) $f(z + w) = f(z)f(w)$
    (b) $f' = f$
    (c) f restricted to the real numbers is the real exponential function.

    This function is denoted e^z and is given by

$$e^z = e^x(\cos y + i \sin y)$$

    where $z = x + iy$.

    Since complex differentiation is beyond the scope of this course the proof will be omitted.

12. Corollary (Euler's Formula)

$$\begin{aligned} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \end{aligned}$$

    from which follows:

$$\begin{aligned} \cos t &= \frac{e^{it} + e^{-it}}{2} \\ \sin t &= \frac{e^{it} - e^{-it}}{2i} \end{aligned}$$

    motivating the following definition:

13. Definition.

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

14. Corollary.

$$\begin{aligned} \cosh x &= \cos ix \\ \sinh x &= -i \sinh ix \end{aligned}$$

15. Corollary (Osborn's rule)

    A trigonometric identity involving only integer powers of *sin* and cos can be converted into a hyperbolic identity by changing every sin to sinh and cos to cosh and switching the sign of every term containing a product of $2, 6, 10, \ldots$ sin's.

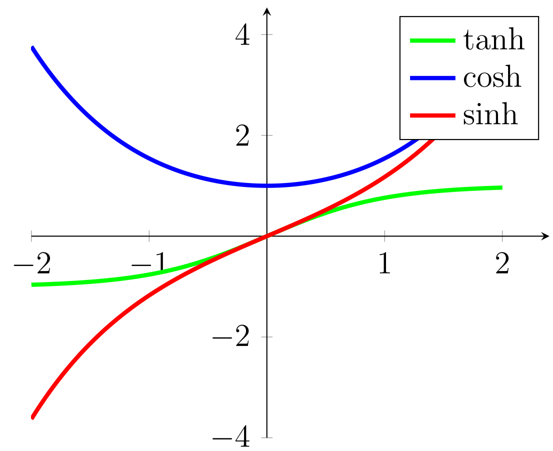
Graphs – Trigonometric and Hyperbolic Functions

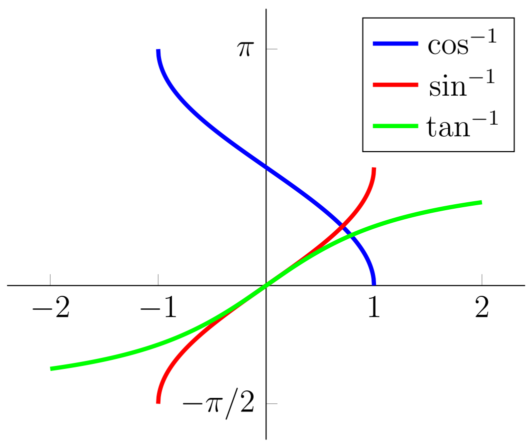
Prof. Philip Pennance[3] -Version: January 16, 2017

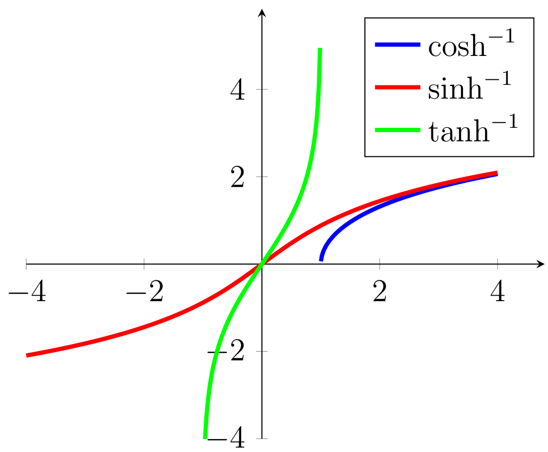
1. (a) $\sinh^{-1}(x) = \ln\left[x + \sqrt{x^2+1}\right]$

   (b) $\cosh^{-1}(x) = \ln\left[x + \sqrt{x^2-1}\right]$

   (c) $\tanh^{-1}(x) = \ln\left[\dfrac{1+x}{1-x}\right]$







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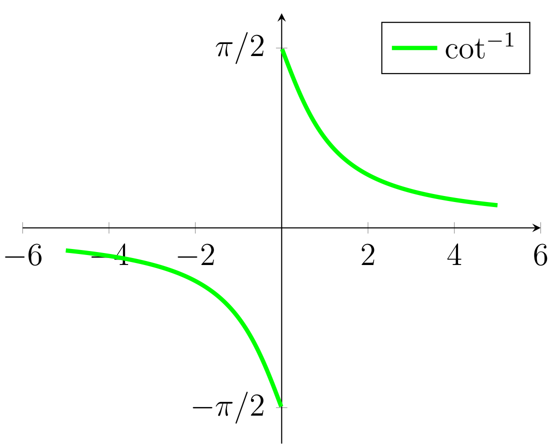
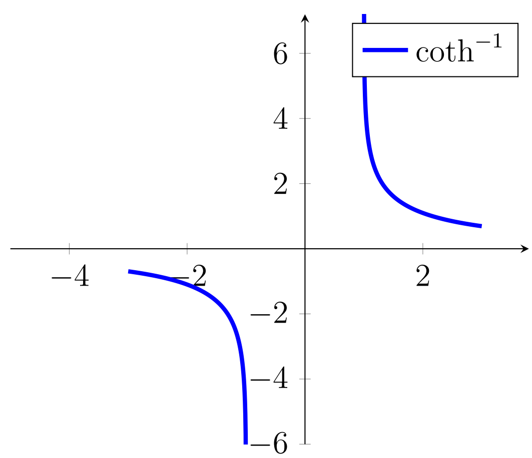




[3] https://pennance.us

Hyperbolic and Trigonometric Integrals

Philip Pennance[4] Version: – February 2019.

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c, \quad |x| < 1$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c, \quad |x| < 1$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c, \quad x \in \mathbb{R}$$
$$\int \frac{-1}{1+x^2}\,dx = \cot^{-1} x + c, \quad x \in \mathbb{R}$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + c, \quad x > 1$$
$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c, x \in \mathbb{R}$$
$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + c, \quad |x| < 1$$
$$\int \frac{1}{1-x^2}\,dx = \coth^{-1} x + c, \quad |x| > 1$$

$$\int \cos^{-1} x\,dx = x\cos^{-1}(x) - \sqrt{1-x^2} + c$$
$$\int \sin^{-1} x\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + c$$
$$\int \tan^{-1} x\,dx = x\tan^{-1}(x) - \frac{1}{2}\log(x^2+1) + c$$

$$\int \cosh^{-1} x\,dx = x\cosh^{-1}(x) - \sqrt{x^2-1} + c$$
$$\int \sinh^{-1} x\,dx = x\sinh^{-1}(x) + \sqrt{1-x^2} + c$$
$$\int \tanh^{-1} x\,dx = x\tanh^{-1}(x) + \frac{1}{2}\log(1-x^2) + c$$

$$\int \cos x\,dx = \sin x + c$$
$$\int \sin x\,dx = -\cosh x + c$$
$$\int \tan x\,dx = -\log|\cos x| + c$$
$$\int \cot x\,dx = \log|\sin x| + c$$
$$\int \sec x\,dx = \log(\sec(x) + \tan(x)) + c$$
$$\int \csc x\,dx = -\log(\cot(x) + \csc(x)) + c$$
$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \cosh x\,dx = \sinh x + c$$
$$\int \sinh x\,dx = \cosh x + c$$
$$\int \tanh x\,dx = \log|\cosh x| + c$$
$$\int \coth x\,dx = \log|\sinh x| + c$$
$$\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + c$$
$$\int \operatorname{csch} x\,dx = \log\left(\tanh\left(\frac{x}{2}\right)\right) + c$$
$$\int \operatorname{sech}^2 x\,dx = \tanh x + c$$

[4] https://pennance.us

The Substitutions $t = \tan(x/2)$ and $t = \tanh(x/2)$

Philip Pennance[5] Version: – February 5, 2022.

1. Claim.

   If
   $$t = \tan\frac{x}{2},$$
   then
   $$\begin{aligned} dx &= \frac{2}{1+t^2}\,dt \\ \cos x &= \frac{1-t^2}{1+t^2} \quad (1) \\ \sin x &= \frac{2t}{1+t^2} \quad (2) \\ \tan x &= \frac{2t}{1-t^2}. \quad (3) \end{aligned}$$

   Proof of (1) .

   Since $x = 2\tan^{-1}(t)$
   $$\begin{aligned} \cos x &= \cos\left(2\cdot\frac{x}{2}\right) \\ &= 2\cos^2\left(\frac{x}{2}\right) - 1 \\ &= \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1 \\ &= \frac{2}{1+t^2} - 1 \\ &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

2. Proof of (2) .
   $$\begin{aligned} \sin x &= \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\cos^2 x + \sin^2 x} \\ &= \frac{2\tan\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} \end{aligned}$$

3. Claim.

   If
   $$t = \tanh\frac{x}{2},$$
   then
   $$\begin{aligned} dx &= \frac{2}{1-t^2}\,dt \\ \cosh x &= \frac{1+t^2}{1-t^2} \quad (4) \\ \sinh x &= \frac{2t}{1-t^2} \quad (5) \\ \tanh x &= \frac{2t}{1+t^2}. \quad (6) \end{aligned}$$

   Proof of (4) .

   Since $x = 2\tanh^{-1}(t)$
   $$\begin{aligned} \cosh x &= \cosh\left(2\cdot\frac{x}{2}\right) \\ &= 2\cosh^2\left(\frac{x}{2}\right) - 1 \\ &= \frac{2}{\operatorname{sech}^2\left(\frac{x}{2}\right)} - 1 \\ &= \frac{2}{1-t^2} - 1 \\ &= \frac{1+t^2}{1-t^2}. \end{aligned}$$

4. Proof of (5) .
   $$\begin{aligned} \sinh x &= \frac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2 x - \sinh^2 x} \\ &= \frac{2\tanh\dfrac{x}{2}}{1-\tanh^2\dfrac{x}{2}} \end{aligned}$$

[5] https://pennance.us

5. Examples

Evaluate $\displaystyle\int \sec x\, dx$

The substitution $t = \tan \frac{x}{2}$ yields

$$\begin{aligned}\int \sec x\, dx &= \int \frac{2}{1-t^2}\, dt \\ &= 2\tanh^{-1}(t) \\ &= 2\tanh^{-1}(\tan \frac{x}{2}) + C\end{aligned}$$

Evaluate $\displaystyle\int \operatorname{sech} x\, dx$

The substitution $t = \tanh \frac{x}{2}$ yields

$$\begin{aligned}\int \operatorname{sech} x\, dx &= \int \frac{2}{1+t^2}\, dt \\ &= 2\tan^{-1}(t) \\ &= 2\tan^{-1}(\tanh \frac{x}{2}) + C\end{aligned}$$

6. Example

Evaluate $\displaystyle\int \frac{1}{1+\sin x}\, dx$

Let $t = \tan(x/2)$. Then

$$\begin{aligned}\int \frac{1}{1+\sin x}\, dx &= \int \frac{1}{1+\left[\dfrac{2t}{1+t^2}\right]} \cdot \frac{2\, dt}{1+t^2} \\ &= \int \frac{2\, dt}{1+2t+t^2} \\ &= \int \frac{2\, dt}{(1+t)^2} \\ &= \frac{-2}{1+t} + C \\ &= \frac{-2}{\tan(x/2)+1} + C\end{aligned}$$

Evaluate $\displaystyle\int \frac{1}{1+\sinh x}\, dx$

Let $t = \tanh(x/2)$. Then

$$\begin{aligned}\int \frac{1}{1+\sinh x}\, dx &= \int \frac{1}{1+\left[\dfrac{2t}{1-t^2}\right]} \cdot \frac{2\, dt}{1-t^2} \\ &= \int \frac{2\, dt}{1+2t-t^2} \\ &= -\int \frac{2\, dt}{(t-1)^2-2} \\ &= \sqrt{2}\tanh^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + C \\ &= \sqrt{2}\tanh^{-1}\left(\frac{\tanh(x/2)-1}{\sqrt{2}}\right) + C\end{aligned}$$

7. Geometric Interpretation.

   If $t = \tan\dfrac{x}{2}$, then

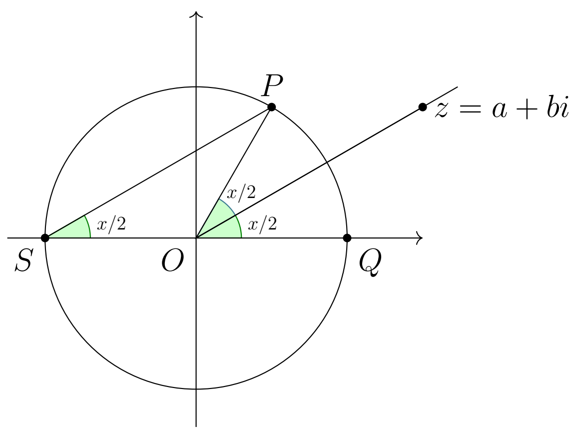
$$dx = \frac{2}{1-t^2}\,dt$$

$$\cos x = \frac{1+t^2}{1+t^2} \tag{7}$$

$$\sin x = \frac{2t}{1+t^2} \tag{8}$$

$$\tan x = \frac{2t}{1+t^2}. \tag{9}$$

   Geometric Proof:



   Let $z = a+bi$ with $b \neq 0$. (see diagram above). If $\arg z = x/2$. Then $t = \tan(x/2) = b/a$. Since squaring a complex number doubles the argument, the point $P = \dfrac{z^2}{|z^2|}$ lies on the unit circle at an angle x. Hence

$$\begin{aligned} P &= \cos x + i\sin x \\ &= \frac{z^2}{|z^2|} \\ &= \frac{a^2-b^2+2abi}{a^2+b^2} \\ &= \frac{1-t^2+2ti}{1+t^2} \end{aligned}$$

   Equating the real and imaginary parts yields:

$$\begin{aligned} \cos x &= \frac{1-t^2}{1+t^2} \\ \sin x &= \frac{2t}{1+t^2} \\ \tan x &= \frac{2t}{1-t^2} \end{aligned}$$

   and so all trigonometric functions can be written in terms of t.

8. Remark.

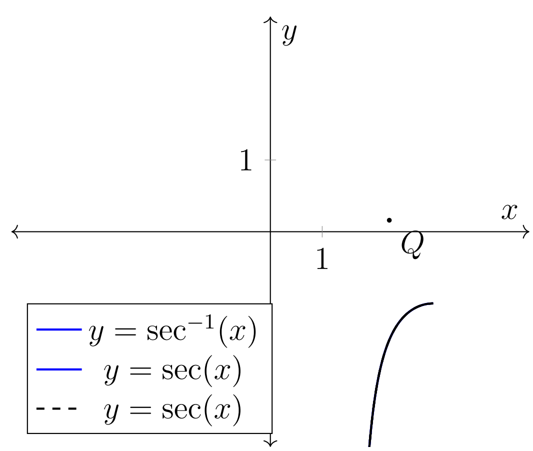
   The substitution $t = \tan(x/2)$ is related to a rational parameterization of the unit circle, used by Euclid, to generate Pythagorean triples. In this formulation, the point z is restricted to lattice points. Further details will be provided in a future document.

9. Remark.

   By Proposition 20 of Euclid's Elements Book III, the arc QP of the unit circle subtends an angle $x/2$ at the point $S = (-1, 0)$ (see diagram). It follows (exercise) that

$$\begin{aligned} \tan\frac{x}{2} &= \frac{\sin x}{1+\cos x} \\ &= \frac{1-\cos x}{\sin x} \end{aligned}$$

10.



Exercises

Prof. Philip Pennance [6] Version: – February 2019.

1. Discuss carefully how the hyperbolic cosine and sine function may be defined by considering the area of a suitable "hyperbolic sector" Hence show that

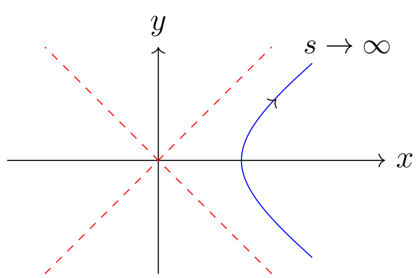
$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

You may assume that
$\int \sqrt{x^2-1} \quad dx =$
$\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\log(x + \sqrt{x^2-1})$

2. Find a parametrization of the left branch of the rectangular hyperbola

$$x^2 - y^2 = 1$$

where $x \leq 1$.
Hint: $\alpha(t) = (\cosh t, \sinh t)$ is a parametrization of the right branch (shown below).



3. Prove the following identities:

   (a) $\sinh(-x) = -\sinh x$
   (b) $\cosh(-x) = -\cosh x$
   (c) $\cosh^2(x) - \sinh^2(x) = 1$
   (d) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
   (e) $\sinh(x + y) = \sinh x \cosh(y) + \cosh x \sinh(y)$
   (f) $\cosh(x + y) = \cosh x \cosh(y) + \sinh x \sinh(y)$

4. Prove formulae for the derivatives of each of the following functions:

   (a) $\sinh x$
   (b) $\cosh x$
   (c) $\tanh x$
   (d) $\operatorname{csch} x$
   (e) $\operatorname{sech} x$
   (f) $\coth x$

5. Prove formulae for:

   (a) $\cosh^{-1} x$
   (b) $\sinh^{-1} x$
   (c) $\tanh^{-1} x$

6. Prove the identities:

   (a) $\cosh x - \sinh x = e^{-x}$
   (b) $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

7. Differentiate:

   (a) $\cosh(\ln x)$
   (b) $\sinh(1 + x^2)$
   (c) $\cosh^{-1}(3x + 5)$
   (d) $\sinh^{-1}(4x^2 - 5)$
   (e) $\tanh^{-1}(e^{2x^3})$
   (f) $\displaystyle\int_0^{\sinh x} e^{x^2}\, dx$

8. Find the integrals:

   (a) $\displaystyle\int \cosh(3x + 4)\, dx$
   (b) $\displaystyle\int \tanh(5x - 7)\, dx$
   (c) $\displaystyle\int \frac{\cosh x}{3 + \sinh x}\, dx$
   (d) $\displaystyle\int \frac{1}{\sqrt{9 + x^2}}\, dx$
   (e) $\displaystyle\int \frac{\sinh \pi x}{\cosh \pi x}\, dx$

[6] https://pennance.us